MATH 12002 - CALCULUS I
§2.2: Differentiability, Graphs, and Higher Derivatives

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Differentiability

The process of finding a derivative is called **differentiation**, and we define:

Definition

Let $y = f(x)$ be a function and let a be a number.
We say f is **differentiable** at $x = a$ if $f'(a)$ exists.

What does this mean in terms of the *graph* of f?

- If $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists, then $f(a)$ must be **defined**.
- Since the denominator is approaching 0, in order for the limit to exist, the numerator must also approach 0; that is,

$$\lim_{h \to 0} (f(a+h) - f(a)) = 0.$$

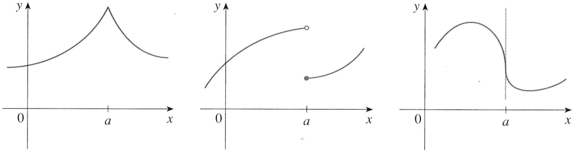
Hence $\lim_{h \to 0} f(a+h) = f(a)$, and so $\lim_{x \to a} f(x) = f(a)$, meaning f must be **continuous at** $x = a$.

Differentiability

But being continuous at a is not enough to make f differentiable at a.
Differentiability is "continuity plus."
The "plus" is *smoothness*: the graph cannot have a sharp "corner" at a.
The graph also cannot have a vertical tangent line at $x = a$:
the slope of a vertical line is not a real number.
Hence, in order for f to be differentiable at a, the graph of f must

1. be continuous at a,
2. be smooth at a, i.e., no sharp corners, and
3. not have a vertical tangent line at $x = a$.

The figure below illustrates how a function can fail to be differentiable:



(a) A corner (b) A discontinuity (c) A vertical tangent

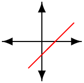
§2.2 Figure 7

Graph of the Derivative

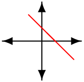
We have seen that certain features of the graph of the function f determine where the derivative function f' is defined.
This is not the only relationship between the graphs of f and f', however.
Recall that $f'(a)$ is the *slope* of the tangent line to f at $x = a$.
In fact, we consider this to be the slope of the graph of f at $x = a$.

Recall what we know about a line and the *sign* of its slope:

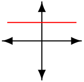
- The slope is positive when the line is “going uphill” as x increases.



- The slope is negative when the line is “going downhill” as x increases.



- The slope is zero when the line is horizontal.



Graph of the Derivative

Similarly, for a general (differentiable) function f, we have the following:

- The slope of the tangent line is *positive*
  when the graph of f is "going uphill" (that is, f is *increasing*).
- The slope of the tangent line is *negative*
  when the graph of f is "going downhill" (that is, f is *decreasing*).

Thus we have the following relationship between the graphs of f and f'.

$$\begin{aligned} f \text{ increasing (graph uphill)} &\leftrightarrow f' \text{ positive (graph of } f' \text{ above } x\text{-axis)} \\ f \text{ decreasing (graph downhill)} &\leftrightarrow f' \text{ negative (graph of } f' \text{ below } x\text{-axis)} \end{aligned}$$

Graph of the Derivative

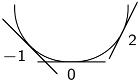
Another important feature of the graph of f is its *concavity*; that is, whether it is “curving upward”

CONCAVE UP

or it is “curving downward”

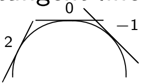
CONCAVE DOWN

If f is *concave up*, slopes of tangent lines *increase* from left to right,



and so f' is increasing on the interval.

If f is *concave down*, slopes of tangent lines *decrease* from left to right,



and so f' is decreasing on the interval.

Graph of the Derivative

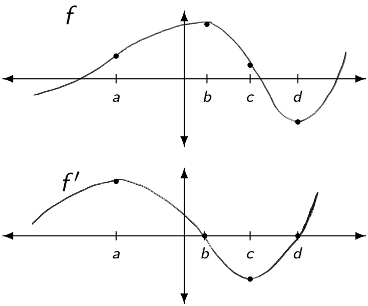
We can now add to the relationship we had before

f increasing $\leftrightarrow$ f' positive
f decreasing $\leftrightarrow$ f' negative

the new relationship

$$f \text{ concave up (curving upward)} \leftrightarrow f' \text{ increasing (graph uphill)}$$
$$f \text{ concave down (curving downward)} \leftrightarrow f' \text{ decreasing (graph downhill)}$$

EXAMPLE:



Higher Derivatives

A "higher derivative" is nothing more than a derivative of a derivative, of a derivative, of a derivative, of a derivative, of a derivative. . .

- The derivative $f'(x)$ of $f(x)$ is also referred to as the **first derivative** of $f(x)$.
- The **second derivative** of $f(x)$ is the derivative of the (first) derivative of $f(x)$, and is denoted $f''(x)$.
- The **third derivative** of $f(x)$ is the derivative of the second derivative of $f(x)$, and is denoted $f'''(x)$.
- The **fourth derivative** of $f(x)$ is the derivative of the third derivative of $f(x)$, and is denoted $f^{(4)}(x)$ (because $f''''(x)$ looks awful).
- You can imagine where it goes from here.

We can interpret higher derivatives in terms of rates of change.

Since $f^{(n+1)}(x)$ is the derivative of $f^{(n)}(x)$, we know that $f^{(n+1)}(x)$ represents the rate of change of $f^{(n)}(x)$ with respect to x.

In particular, we can apply this to position and velocity:

Suppose an object is moving in a straight line.

- $s(t)$ denotes the **position** of the object (in feet, f) at time t (in seconds, s).
- $s'(t)$ is the **rate of change of position** with respect to time at t; that is, $s'(t) = v(t)$ is the **velocity** at time t (in feet per second, f/s).
- $s''(t) = v'(t)$ is the **rate of change of velocity** with respect to time at t; that is, $s''(t) = v'(t) = a(t)$ is the **acceleration** at time t (in feet per second per second, $(f/s)/s$).

Higher Derivatives

The second derivative of f also gives information about the graph of f. Recall that

$$\boxed{\begin{array}{r} f \text{ increasing} \leftrightarrow f' \text{ positive} \\ f \text{ decreasing} \leftrightarrow f' \text{ negative} \end{array}}$$

Applying this to the function f', we have

$$\begin{array}{r} f' \text{ increasing} \leftrightarrow (f')' \text{ positive} \\ f' \text{ decreasing} \leftrightarrow (f')' \text{ negative.} \end{array}$$

Recalling also that

$$\boxed{\begin{array}{r} f \text{ concave up} \leftrightarrow f' \text{ increasing} \\ f \text{ concave down} \leftrightarrow f' \text{ decreasing} \end{array}}$$

and observing that $(f')'$ is the second derivative, f'', of f, we get

$$\boxed{\begin{array}{rcccc} f \text{ concave up} & \leftrightarrow & f' \text{ increasing} & \leftrightarrow & f'' \text{ positive} \\ f \text{ concave down} & \leftrightarrow & f' \text{ decreasing} & \leftrightarrow & f'' \text{ negative} \end{array}}$$