MATH 12002 - CALCULUS I
§2.7: Related Rates Part 3: More Examples

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Examples

Example 1

At noon, ship A is 100 *km west of ship B. Ship A is sailing south at* 35 *kilometers per hour (km/h) and ship B is sailing north at* 25 *km/h. How fast is the distance between the ships changing at* 4 *pm?*

Solution

Let

- $A =$ *distance from starting point to ship A,*
- $B =$ *distance from starting point to ship B,*
- $Z =$ *distance between the ships.*

Given: $\frac{dA}{dt} = 35$ *km/h and* $\frac{dB}{dt} = 25$ *km/h.*

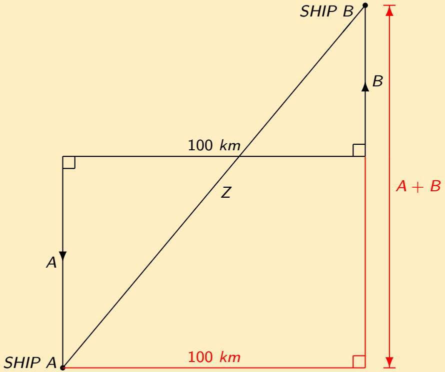
Want: $\frac{dZ}{dt}$ *at* 4 *pm.*

We need to relate A, B, and Z and then take derivatives with respect to time in order to relate $\frac{dA}{dt}$, $\frac{dB}{dt}$, *and* $\frac{dZ}{dt}$.

[Continued →]

Example 1 Solution [continued]

We have the following situation at a given time after noon:



By the Pythagorean Theorem, $100^2 + (A+B)^2 = Z^2$.

[Continued →]

Examples

Example 1 Solution [continued]

We now have $Z^2 = 100^2 + (A + B)^2$, *and taking derivatives with respect to time yields*

$$2Z \cdot \frac{dZ}{dt} = 2(A + B)\left[\frac{dA}{dt} + \frac{dB}{dt}\right].$$

Hence

$$\frac{dZ}{dt} = \frac{A + B}{Z}\left[\frac{dA}{dt} + \frac{dB}{dt}\right].$$

Finally, at 4 *pm,* $A = (35)(4) = 140\ km$, $B = (25)(4) = 100\ km$, *and*

$$Z = \sqrt{100^2 + (140 + 100)^2} = \sqrt{100^2 + 240^2} = 260\ km,$$

and so

$$\frac{dZ}{dt} = \frac{140 + 100}{260}(35 + 25) = \frac{720}{13} \approx \boxed{55.4\ km/h.}$$

Examples

Example 2

A plane flying with a constant speed of 210 miles per hour passes over a ground radar station at an altitude of 2 miles and climbs at an angle of 45°. How fast is the distance from the plane to the radar station increasing 2 minutes later?

Solution

Let

- $p =$ *distance of plane from the point where it passed over the station,*
- $z =$ *distance from the plane to the station.*

Given: $\frac{dp}{dt} = 210$ *mph.*

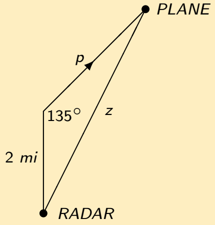
Want: $\frac{dz}{dt}$ *after 2 minutes.*

We need to relate p and z and then take derivatives with respect to time in order to relate $\frac{dp}{dt}$ *and* $\frac{dz}{dt}$.

[Continued →]

Example 2 Solution [continued]

We have the following situation at a given time:



We use the Law of Cosines to relate p *and* z*, and this says*

$$\begin{aligned} z^2 &= 2^2 + p^2 - 2(2)(p)\cos 135^\circ \\ &= 4 + p^2 - 4p \cdot (-\sqrt{2}/2) \\ &= 4 + p^2 + 2\sqrt{2}p. \end{aligned}$$

[Continued →]

Example 2 Solution [continued]

We have $z^2 = p^2 + 2\sqrt{2}p + 4$, *and taking derivatives with respect to time, we obtain*

$$2z\frac{dz}{dt} = 2p\frac{dp}{dt} + 2\sqrt{2}\frac{dp}{dt},$$

and so

$$\begin{aligned} \frac{dz}{dt} &= \frac{2p\frac{dp}{dt} + 2\sqrt{2}\frac{dp}{dt}}{2z} \\ &= \frac{p+\sqrt{2}}{z} \cdot \frac{dp}{dt}. \end{aligned}$$

We know $\frac{dp}{dt} = 210$ *mph, and so we now need to determine the values of* p *and* z *two minutes after the plane passes over the station.*

[Continued →]

Examples

Example 2 Solution [continued]

Since the plane is flying at a speed of 210 *mph,* 2 *minutes (or* $\frac{1}{30}$ *hour) after it passes over the station,*

$$p = 210 \cdot \tfrac{1}{30} = 7 \ \textit{miles},$$

and

$$z = \sqrt{7^2 + 2\sqrt{2}(7) + 4} = \sqrt{53 + 14\sqrt{2}} \ \textit{miles}.$$

Therefore,

$$\frac{dz}{dt} = \frac{7 + \sqrt{2}}{\sqrt{53 + 14\sqrt{2}}} \cdot 210 \approx \boxed{207 \ \textit{mph.}}$$