MATH 4/51001
COSETS

Let G be a group and let H be a subgroup of G. We defined equivalence relations $\sim_r$ and $\sim_\ell$ on G as follows:

For a, b in G, $a \sim_r b$ if and only if $ab^{-1} \in H$.
For a, b in G, $a \sim_\ell b$ if and only if $a^{-1}b \in H$.

The equivalence classes of G with respect to $\sim_r$ are the *right cosets* of H in G and the equivalence classes of G with respect to $\sim_\ell$ are the *left cosets* of H in G.

RIGHT COSETS

We proved the following results for *right* cosets in class.

Theorem 1. *Let $H \leqslant G$ and let $a \in G$. The equivalence class of a with respect to $\sim_r$ is*

$$Ha = \{ha \mid h \in H\}.$$

Thus Ha is the right coset of H containing a.

Theorem 2. *Let $H \leqslant G$ and let $a, b \in G$. The following are equivalent:*

i. $Ha \cap Hb \neq \emptyset$
ii. $Ha = Hb$
iii. $b \in Ha$
iv. $a \in Hb$
v. $b = ha$ *for some* $h \in H$
vi. $a = kb$ *for some* $k \in H$
vii. $ba^{-1} \in H$
viii. $ab^{-1} \in H$.

LEFT COSETS

Prove the following analogous results for *left* cosets as an exercise.

Theorem 1′. *Let $H \leqslant G$ and let $a \in G$. The equivalence class of a with respect to $\sim_\ell$ is*

$$aH = \{ah \mid h \in H\}.$$

Thus aH is the left coset of H containing a.

Theorem 2′. *Let $H \leqslant G$ and let $a, b \in G$. The following are equivalent:*

i. $aH \cap bH \neq \emptyset$
ii. $aH = bH$
iii. $b \in aH$
iv. $a \in bH$
v. $b = ah$ *for some* $h \in H$
vi. $a = bk$ *for some* $k \in H$
vii. $a^{-1}b \in H$
viii. $b^{-1}a \in H$.