The following are the basic facts on transpositions and even and odd permutations in $S_n$. The proofs are in §2.3 of the text. Recall that a *transposition* is a cycle of length 2.

- A cycle of length $k$ can be written as a product of $k - 1$ transpositions:
  $$(a_1, a_2, \ldots, a_{k-1}, a_k) = (a_{k-1}, a_k)(a_{k-2}, a_k) \cdots (a_2, a_k)(a_1, a_k).$$

- Every permutation in $S_n$ can be written as a product of cycles, hence also as a product of transpositions. The transpositions will NOT generally be disjoint. For example:
  $$(1, 3, 5, 2, 4, 9)(6, 10, 13)(7, 12, 11, 8) = (4, 9)(2, 9)(5, 9)(3, 9)(1, 9) \cdot (10, 13)(6, 13) \cdot (11, 8)(12, 8)(7, 8).$$

- A permutation can be written either as the product of an even number of transpositions or as the product of an odd number of transpositions, but not both.

- We say an element of $S_n$ is
  - an *even permutation* if it can be written as a product of an even number of transpositions,
  - an *odd permutation* if it can be written as a product of an odd number of transpositions.

- A cycle of length $k$ can be written as the product of $k - 1$ transpositions. Hence (unfortunately)
  - a cycle of odd length is an *even* permutation,
  - a cycle of even length is an *odd* permutation.

- If $\sigma$, $\tau$ are permutations, the number of factors in an expression of $\sigma\tau$ as a product of transpositions is the sum of the numbers for $\sigma$ and $\tau$. We therefore have
  - even permutation · even permutation = even permutation
  - odd permutation · odd permutation = even permutation
  - odd permutation · even permutation = odd permutation.

- Given a permutation $\sigma$ as a product of cycles (not necessarily disjoint), the last two facts allow us to determine the parity of $\sigma$:
  - If $\sigma$ has an even number of cycles of even length, then $\sigma$ is even.
  - If $\sigma$ has an odd number of cycles of even length, then $\sigma$ is odd.

- Alternatively, we can define a sign function on $S_n$ by
  $$\text{sign}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd}. \end{cases}$$

  The results above then say that $\text{sign}(\sigma\tau) = \text{sign}(\sigma) \cdot \text{sign}(\tau)$ for $\sigma, \tau \in S_n$.

- For example, if $\sigma = (1, 2, 3)(5, 7, 8, 9)(4, 6, 2, 5)(3, 7, 6, 8, 4)(1, 2)$, then
  $$\text{sign}(\sigma) = \text{sign}(1, 2, 3) \cdot \text{sign}(5, 7, 8, 9) \cdot \text{sign}(4, 6, 2, 5) \cdot \text{sign}(3, 7, 6, 8, 4) \cdot \text{sign}(1, 2)$$
  $$= (+1)(-1)(-1)(+1)(-1) = -1.$$  
  Hence $\text{sign}(\sigma) = -1$ and therefore $\sigma$ is an odd permutation.