MATH 41001/51001
FINAL EXAM

Prof. D. L. White
December 14, 2006 NAME:______________________________

Write clearly and show all work. Put your name on each page. Time allowed is 2 hours 15 minutes.

1. (30 points) Using complete sentences, define the term or state the theorem.

   (a) Lagrange's Theorem

   (b) Group Homomorphism

   (c) Isomorphic Groups

   (d) Alternating Group

   (e) Normal Subgroup

   (f) First (or Fundamental) Isomorphism Theorem

NAME:______________________________

2. (20 points) Let

$$G = \langle g \rangle = \{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8, g^9, g^{10}, g^{11}, g^{12}, g^{13}, g^{14}, g^{15}, g^{16}, g^{17}, g^{18}, g^{19}\}$$

be a cyclic group of order 20.

(a) Find the orders of the elements g^6, g^{10}, g^{12}, and g^{15} of G.

(b) Find all generators of G.

(c) Find all nontrivial, proper subgroups of G and list the elements of each.

NAME:______________________________

3. (20 points) Let

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \;\middle|\; a, b, d \in \mathbb{R} \text{ and } ad \neq 0 \right\} \subseteq \mathrm{GL}_2(\mathbb{R})$$

and

$$N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\} \subseteq G.$$

(a) Show that G is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

(b) Show that N is a *normal* subgroup of G. (You may assume N is a subgroup of G.)

It may help to recall that $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & -\frac{b}{ad} \\ 0 & \frac{1}{d} \end{bmatrix}$.

NAME:______________________________

4. (20 points) Let G be a group and fix an element a of G. Show that the map $\varphi : G \to G$, defined by $\varphi(g) = aga^{-1}$ for all $g \in G$, is an isomorphism.

NAME:______________________________

5. (20 points) Let G be a group of order pq, where p and q are primes. Show that every proper subgroup of G is cyclic.

NAME:______________________________

6. (20 points) Show that if G is an abelian group and N is a subgroup of G, then $N \trianglelefteq G$ and G/N is abelian.

NAME:______________________________

7. (20 points) Let $G = \mathrm{GL}_2(\mathbb{R})$ be the group of non-singular 2×2 matrices with real number entries under matrix multiplication, and let

$$N = \{X \in G \mid \det X = 1 \text{ or } \det X = -1\}.$$

Show that $G/N \cong \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the group of positive real numbers under multiplication. [Hint: Consider the map $\varphi : G \to \mathbb{R}^+$ given by $\varphi(X) = |\det X|$.]