MATH 4/51001
Homework #9
Due: Wednesday, November 15

Reading:

For Monday, November 13: §3.8

Problems to turn in:

§**3.7:** 4, 6, 7(b,d,f), 8.

I. Show that each of the following functions from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$ is a group homomorphism and determine the kernel and image:
   (a) $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by $\varphi([a]) = 3[a]$ for all $[a]$ in $\mathbb{Z}_{12}$;
   (b) $\psi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by $\psi([a]) = 5[a]$ for all $[a]$ in $\mathbb{Z}_{12}$.

II. For groups G_1 and G_2, let $\pi_1 : G_1 \times G_2 \to G_1$ be defined by $\pi_1((g_1, g_2)) = g_1$ and let $\pi_2 : G_1 \times G_2 \to G_2$ be defined by $\pi_2((g_1, g_2)) = g_2$. Show that π_1 and π_2 are group homomorphisms and determine the kernel and image of each.

III. Let G and H be finite groups and let $\varphi : G \to H$ be a group homomorphism.
   (a) Show that if $|G|$ is a prime, then φ is either one-to-one or the trivial homomorphism.
   (b) Show that if $|H|$ is a prime, then φ is either onto or the trivial homomorphism.

IV. Let $\varphi : G \to H$ be a homomorphism of groups. Fix an element g of G and let $\varphi(g) = h \in H$. Show that the preimage $\varphi^{-1}(h)$ of h under φ is the set

$$\varphi^{-1}(h) = \{kg \,|\, k \in \ker \varphi\}.$$

Notes:

- For each map in §3.7 #7, either *prove* that the map is a homomorphism or give a specific counter-example to show that the map is *not* a homomorphism.
- On §3.7 #8, $\theta\phi$ denotes the composite $\theta \circ \phi$.
- On #II, recall §3.3 #11.
- On #IV, recall that the preimage $\varphi^{-1}(h)$ of h under φ is *defined* to be

$$\varphi^{-1}(h) = \{a \in G \,|\, \varphi(a) = h\}.$$

  Hence show that for $a \in G$, $\varphi(a) = h = \varphi(g)$ if and only if $a = kg$ for some $k \in \ker \varphi$.

Problems to be aware of:

§3.7: 3, 5, 9.