MATH 41002/51002
FINAL EXAM

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Write clearly and show all work. Put your name on each page. Time allowed is 2 hours 15 minutes.

1. (30 points) Using complete sentences, define the term or state the theorem.

   (a) Division Algorithm (for Polynomials)

   (b) Rational Root Test

   (c) Ideal (of a Ring)

   (d) Degree of a Field Extension

   (e) Splitting Field

   (f) Constructibility Theorem

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2. (15 points) Let I be an ideal of a ring R. Show that if I contains a unit of R, then $I = R$.

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3. (15 points) Let R and S be commutative rings and $\varphi : R \to S$ a surjective ring homomorphism. Let J be an ideal of S. Show that if the preimage $I = \{a \in R \mid \varphi(a) \in J\}$ of J under φ is a prime ideal of R, then J is a prime ideal of S.

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4. (15 points) Let V and W be vector spaces over a field F and let $\varphi : V \to W$ be a linear transformation. Show that $\ker \varphi$ is a subspace of V and $\text{Im } \varphi$ is a subspace of W.

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5. (15 points) Let V and W be vector spaces over a field F and let $\varphi : V \to W$ be a linear transformation. Let

$$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$$

be a subset of V. Show that if φ is injective and

$$T = \{\varphi(\boldsymbol{v}_1), \varphi(\boldsymbol{v}_2), \ldots, \varphi(\boldsymbol{v}_n)\}$$

spans $\operatorname{Im} \varphi$, then S spans V.

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6. (15 points) Let $K \subseteq L \subseteq F$ be fields. Show that if $\alpha \in F$ is algebraic over K, then

$$[L(\alpha) : L] \leqslant [K(\alpha) : K].$$

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7. (15 points) Let α_1, α_2, α_3 be real numbers such that $\alpha_i^2 \in \mathbb{Q}$ for each i, and let $F = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$. Show that $\sqrt[3]{7}$ is NOT an element of F. [Hint: Consider the field $\mathbb{Q}(\sqrt[3]{7})$ and degrees of extensions.]

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8. (15 points) Let $f(x) = x^3 + x + 1 \in \mathbb{Z}_2[x]$ and let α be a root of $f(x)$ in an extension of $\mathbb{Z}_2$. Let $F = \mathbb{Z}_2(\alpha)$, with basis $B = \{1, \alpha, \alpha^2\}$ over $\mathbb{Z}_2$. (You need not prove this.)

   (a) Write all of the elements of $F = \mathbb{Z}_2(\alpha)$ in terms of the basis B.

   (b) Write each of the following elements of F in terms of the basis B:

      i. $\alpha^6 + \alpha^4 + \alpha^2 + 1$

      ii. $(\alpha^2 + \alpha)(\alpha^2 + 1)$

      iii. $(\alpha + 1)^4$.

   (c) Find a generator for the multiplicative group $F^\times$ of F.

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9. (15 points) Let $K = \mathbb{Z}_2$ and $F = \mathbb{Z}_2(\beta)$ the field of 8 elements, where β is a root of $x^3+x^2+1 \in \mathbb{Z}_2[x]$.

   (a) Factor $x^8 - x$ as a product of irreducible polynomials over $F = \mathbb{Z}_2(\beta)$.

   (b) Factor $x^8 - x$ as a product of irreducible polynomials over $K = \mathbb{Z}_2$.