

(46)

Theorem: Let $B = \{\bar{w}_1, \dots, \bar{w}_k\}$ be an orthogonal set of vectors in V and let $W = \text{sp}(B)$. For any vector \bar{v} in V , we have
(i) $\text{proj}(\bar{v}, W)$ is the closest vector to \bar{v} in W that is, if $\bar{w} \in W$ then $\|\bar{v} - \text{proj}(\bar{v}, W)\| \leq \|\bar{v} - \bar{w}\|$,

(ii) $\text{proj}(\bar{v}, W) = \bar{v}$ if and only if $\bar{v} \in W$,

(iii) $\bar{v}' = \bar{v} - \text{proj}(\bar{v}, W)$ is orthogonal to every vector in W , in particular, \bar{v}' is orthogonal to $\bar{w}_1, \dots, \bar{w}_k$.

Proof: See text, Problems 7.25, 7.30. \square

This theorem underlies the Gram-Schmidt Orthogonalization.

Theorem: Let $B = \{\bar{v}_1, \dots, \bar{v}_n\}$ be a basis for V and construct

$$\bar{w}_1 = \bar{v}_1$$

$$\bar{w}_2 = \bar{v}_2 - \frac{\langle \bar{v}_2, \bar{w}_1 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle} \bar{w}_1 = \bar{v}_2 - \text{proj}(\bar{v}_2, \text{sp}\{\bar{w}_1\})$$

$$\bar{w}_3 = \bar{v}_3 - \frac{\langle \bar{v}_3, \bar{w}_1 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle} \bar{w}_1 - \frac{\langle \bar{v}_3, \bar{w}_2 \rangle}{\langle \bar{w}_2, \bar{w}_2 \rangle} \bar{w}_2 = \bar{v}_3 - \text{proj}(\bar{v}_3, \text{sp}\{\bar{w}_1, \bar{w}_2\})$$

\vdots

$$\bar{w}_n = \bar{v}_n - \frac{\langle \bar{v}_n, \bar{w}_1 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle} \bar{w}_1 - \dots - \frac{\langle \bar{v}_n, \bar{w}_{n-1} \rangle}{\langle \bar{w}_{n-1}, \bar{w}_{n-1} \rangle} \bar{w}_{n-1}$$

$$= \bar{v}_n - \text{proj}(\bar{v}_n, \text{sp}\{\bar{w}_1, \dots, \bar{w}_{n-1}\}).$$

Then $B' = \{\bar{w}_1, \dots, \bar{w}_n\}$ is an orthogonal basis for V . \square

[Observe that if we define $W_i = \text{sp}\{\bar{w}_1, \dots, \bar{w}_i\}$ for each i , then $\bar{w}_{i+1} = \bar{v}_{i+1} - \text{proj}(\bar{v}_{i+1}, W_i)$, thus \bar{w}_{i+1} is the component of \bar{v}_{i+1} orthogonal to the subspace W_i .]