MATH 6/71051
EXAM II Solutions

1. *Show that if H is a characteristic subgroup of N and N is a characteristic subgroup of G, then H is a characteristic subgroup of G.*

Proof. Let $H \operatorname{char} N$ and $N \operatorname{char} G$, and let $\varphi \in \operatorname{Aut}(G)$. Since $N \operatorname{char} G$, we have $\varphi(N) = N$, and so the restriction φ_N of φ to N is an automorphism of N. Hence $\varphi(H) = \varphi_N(H) = H$ because $H \operatorname{char} N$. We have $\varphi(H) = H$ for all $\varphi \in \operatorname{Aut}(G)$, so H is a characteristic subgroup of G. ☐

2. *Let G be a group acting transitively on a set A. Show that if there is an element $a \in A$ such that $G_a = \langle 1 \rangle$, then $G_b = \langle 1 \rangle$ for all $b \in A$.*

Proof. Let $a \in A$ with $G_a = \langle 1 \rangle$ and let $b \in A$. Since G acts transitively on A, there is an element $g \in G$ with $b = g \cdot a$, and so $G_b = G_{g \cdot a}$. We then have

$$\begin{aligned} x \in G_b = G_{g\cdot a} &\Leftrightarrow x \cdot (g \cdot a) = g \cdot a \\ &\Leftrightarrow g^{-1} \cdot (x \cdot (g \cdot a)) = a, \text{ by axioms of group action,} \\ &\Leftrightarrow (g^{-1}xg) \cdot a = a, \text{ by axioms of group action,} \\ &\Leftrightarrow g^{-1}xg \in G_a \\ &\Leftrightarrow g^{-1}xg = 1, \text{ since } G_a = \langle 1 \rangle, \\ &\Leftrightarrow x = 1. \end{aligned}$$

Therefore, $G_b = \langle 1 \rangle$ for all $b \in A$. ☐

3. *Show that if the size of each conjugacy class of a group G is at most 2, then $G' \leqslant Z(G)$.*

Proof. Let $g \in G$. Since the size of the conjugacy class of g is at most 2, we have $|G : C_G(g)| \leqslant 2$. Hence $C_G(g) \trianglelefteq G$ and $G/C_G(g)$ is abelian, and so $G' \leqslant C_G(g)$. Therefore, $G' \leqslant C_G(g)$ for all $g \in G$, thus $G' \leqslant \cap_{g \in G} C_G(g) = Z(G)$. ☐

4. *Let G be a group of order $3 \cdot 5 \cdot 7 \cdot 13$. Prove that G is not a simple group.*

Proof. Suppose $|G| = 3 \cdot 5 \cdot 7 \cdot 13$ and G is simple. First, $n_7 \mid 3 \cdot 5 \cdot 13$ and $n_7 \equiv 1 (\text{mod } 7)$, hence $n_7 = 1$ or $n_7 = 15$. Since G is simple, we have $n_7 = 15$, and so if $S \in \operatorname{Syl}_7(G)$, then $|G : N_G(S)| = 15$ and $|N_G(S)| = 7 \cdot 13$.

Let T be a Sylow 13-subgroup of $N_G(S)$. Since $|T| = 13$, we also have $T \in \operatorname{Syl}_{13}(G)$. Since $|N_G(S)| = 7 \cdot 13$ and $13 > 7$, we have $T \trianglelefteq N_G(S)$, and so $N_G(S) \leqslant N_G(T)$. Now $n_{13} \mid 3 \cdot 5 \cdot 7$ and $n_{13} \equiv 1 (\text{mod } 13)$, hence $n_{13} = 1$ or $n_{13} = 3 \cdot 5 \cdot 7$. Since G is simple, we have $n_{13} = 3 \cdot 5 \cdot 7$, hence $|G : N_G(T)| = 3 \cdot 5 \cdot 7$ and $|N_G(T)| = 13$. This contradicts $N_G(S) \leqslant N_G(T)$, and so G cannot be simple. ☐

5. *Let G be a finite group and let P be a Sylow p-subgroup of G. Show the following.*

(a) *If M is any normal p-subgroup of G then $M \le P$.*

(b) *There is a normal p-subgroup N of G that contains all normal p-subgroups of G.*

Proof.

(a) Let $M \trianglelefteq G$ be a p-subgroup. By Sylow's Theorem, $M \leqslant Q$ for some $Q \in \mathrm{Syl}_p(G)$ and $P = gQg^{-1}$ for some $g \in G$. As $M \trianglelefteq G$, we have $M = gMg^{-1} \leqslant gQg^{-1} = P$.

(b) By Sylow's Theorem, $\mathrm{Syl}_p(G)$ consists precisely of the conjugates of P. Hence

$$N = \bigcap_{Q \in \mathrm{Syl_p(G)}} Q = \bigcap_{g \in G} gPg^{-1} = \mathrm{core}_G(P) \trianglelefteq G.$$

By (a), if M is any normal p-subgroup of G, then $M \leqslant N$. □

6. *List all possible sets of invariant factors and elementary divisors for an abelian group of order* $2600 = 2^3 \cdot 5^2 \cdot 13$*, and match the invariant factors and elementary divisors of isomorphic groups.*

Solution. The possible sets of invariant factors for a group of order 2^3 are $\{2,2,2\}$, $\{2^2,2\}$, or $\{2^3\}$. For a group of order 5^2, they are $\{5,5\}$ or $\{5^2\}$, and the only possibility for a group of order 13 is $\{13\}$. There are then 6 distinct abelian groups of order $2600 = 2^3 \cdot 5^2 \cdot 13$, up to isomorphism. The elementary divisors and invariant factors are as follows:

ElementaryDivisors		InvariantFactors
$\{2,2,2,5,5,13\}$	$\longleftrightarrow$	$\{2 \cdot 5 \cdot 13, 2 \cdot 5, 2\} = \{130, 10, 2\}$
$\{2,2,2,5^2,13\}$	$\longleftrightarrow$	$\{2 \cdot 5^2 \cdot 13, 2, 2\} = \{650, 2, 2\}$
$\{2^2,2,5,5,13\}$	$\longleftrightarrow$	$\{2^2 \cdot 5 \cdot 13, 2 \cdot 5\} = \{260, 10\}$
$\{2^2,2,5^2,13\}$	$\longleftrightarrow$	$\{2^2 \cdot 5^2 \cdot 13, 2\} = \{1300, 2\}$
$\{2^3,5,5,13\}$	$\longleftrightarrow$	$\{2^3 \cdot 5 \cdot 13, 5\} = \{520, 5\}$
$\{2^3,5^2,13\}$	$\longleftrightarrow$	$\{2^3 \cdot 5^2 \cdot 13\} = \{2600\}$

□

7. *Let G be a finite group.*

(a) *Show that if M is a maximal subgroup of G, then either $Z(G) \leq M$ or $G' \leq M$.*

(b) *Show that $Z(G) \cap G' \leq \Phi(G)$.*

Proof.

(a) Let M be a maximal subgroup of G and suppose $Z(G)$ is not contained in M. Since $Z(G) \trianglelefteq G$, we have $M < Z(G)M \leqslant G$, and since M is maximal, we have $Z(G)M = G$. Certainly M normalizes M and, since elements of $Z(G)$ commute with all elements of M, $Z(G)$ also normalizes M. Hence the product $G = Z(G)M$ normalizes M, and so $M \trianglelefteq G$.

Since M is a maximal subgroup of G and $M \trianglelefteq G$, we have $|G : M|$ is a prime (see Exam I, #8), hence G/M is abelian. It follows that $G' \leqslant M$. Therefore, either $Z(G) \leqslant M$ or $G' \leqslant M$.

(b) Let M be a maximal subgroup of G. By (a), $Z(G) \leqslant M$ or $G' \leqslant M$. But $Z(G) \cap G' \leqslant Z(G)$ and $Z(G) \cap G' \leqslant G'$, hence $Z(G) \cap G' \leqslant M$ in any case. It follows that $Z(G) \cap G'$ is contained in the intersection of all maximal subgroups; that is, $Z(G) \cap G' \leqslant \Phi(G)$. $\square$

8. *Let G be a finite group. Show that G is nilpotent if and only if whenever p and q are distinct primes and $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, then P centralizes Q.*

Proof.

$\boxed{\Rightarrow}$ Assume G is nilpotent, so that G is the direct product of its Sylow subgroups. Thus if $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ for distinct primes p, q, then all elements of P commute with all elements of Q, i.e., P centralizes Q.

$\boxed{\Leftarrow}$ Let $p_1, p_2, \ldots, p_n$ be the distinct primes dividing $|G|$ and let $P_i \in \mathrm{Syl}_{p_i}(G)$ for each i. Assume P_i centralizes P_j whenever $i \neq j$. We first use induction on n to prove that $P_1 P_2 \cdots P_n$ is a subgroup of G.

Since $P_1 \leqslant G$, the statement is true if $n = 1$. Assume now that $1 < k \leqslant n$ and $P_1 P_2 \cdots P_{k-1} \leqslant G$. For each $i < k$, P_i centralizes P_k and so $P_i \leqslant N_G(P_k)$. Therefore, the subgroup $P_1 P_2 \cdots P_{k-1}$ also normalizes P_k, thus $(P_1 P_2 \cdots P_{k-1}) \cdot P_k \leqslant G$. Hence it follows by induction that $H = P_1 P_2 \cdots P_n$ is a subgroup of G.

Again, since for each $j \neq i$, P_j centralizes P_i, we have $P_i \trianglelefteq H$ for each i. By Lagrange's Theorem, $P_i \cap \prod_{j \neq i} P_j = \langle 1 \rangle$. Hence $H = P_1 \times P_2 \times \cdots \times P_n$. By definition of Sylow subgroup, we have $|H| = |P_1| \cdot |P_2| \cdots |P_n| = |G|$. Hence $G = P_1 \times P_2 \times \cdots \times P_n$, that is, G is a direct product of its Sylow subgroups, and so G is nilpotent. $\square$