

MATH 6/71051  
Homework #7  
**Due: Thursday, October 25**

**Reading:**

For Tues., October 23: §5.1 & §5.4  
For Thurs., October 25: §5.2 & §5.3

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**Problems to turn in:**

**§4.6:** 2, 3.

- I. Let  $G$  be a finite group and  $p$  a prime. Show that the intersection of all Sylow  $p$ -subgroups of  $G$  is a normal subgroup of  $G$ .
- II. Let  $K$  be a normal subgroup of  $G$  and let  $P$  be a Sylow  $p$ -subgroup of  $K$ . Show that if  $P \trianglelefteq K$  then  $P \trianglelefteq G$ .
- III. Let  $P$  be a  $p$ -subgroup of  $G$ . Show that  $P$  is a Sylow  $p$ -subgroup of  $G$  if and only if  $P$  is a Sylow  $p$ -subgroup of  $N_G(P)$ .
- IV. Let  $G$  be a simple group and let  $p$  be a prime. Show that if  $p^2$  divides the order of  $G$ , then  $G$  has no subgroup of index  $p$ .
- V. (a) Show that  $S_6$  has no simple subgroup of index 4 (i.e. order 180).  
(b) Show that a group of order  $180 = 2^2 \cdot 3^2 \cdot 5$  cannot be simple.
- VI. Show that a group of order  $2001 = 3 \cdot 23 \cdot 29$  must contain a normal cyclic subgroup of index 3.
- VII. Let  $G$  be a simple group of order 168.  
(a) How many elements of order 7 does  $G$  have? (Prove your answer!).  
(b) Show that  $G$  is isomorphic to a subgroup of  $A_8$ .
- VIII. Show that a group of order  $10000 = 2^4 \cdot 5^4$  cannot be simple.

**Notes:**

- We will not cover §5.3 explicitly in class. It contains a useful list of small groups.
  - There *is* a simple group of order 168, namely  $\text{PSL}_2(\mathbb{F}_7) = \text{SL}_2(\mathbb{F}_7)/Z(\text{SL}_2(\mathbb{F}_7))$ , where  $\mathbb{F}_7 = \mathbb{Z}_7$  is the field of 7 elements, or  $\text{GL}_3(\mathbb{F}_2)$ , where  $\mathbb{F}_2 = \mathbb{Z}_2$  is the field of 2 elements. Moreover,  $\text{PSL}_2(\mathbb{F}_7) \cong \text{GL}_3(\mathbb{F}_2)$ , and this is the *only* simple group of order 168.
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**Problems to be aware of:**

§4.5: **1**, 2, 33.  
§4.6: 1, 4.

[Problems in **bold** contain results we will assume and use in the course.]