MATH 6/71051
Homework #8
Selected Solutions

#I. *Let G be a finite group and p a prime. If N is a normal subgroup of G and P is a Sylow p-subgroup of G, then*

(a) *PN/N is a Sylow p-subgroup of G/N, and*

(b) *$P \cap N$ is a Sylow p-subgroup of N.*

Proof. For an integer ℓ, let ℓ_p denote the p-part of ℓ; i.e., if $\ell = p^r s$ and $p \nmid s$, then $\ell_p = p^r$. Thus a p-subgroup Q of a finite group K is a Sylow p-subgroup of K if and only if $|Q| = |K|_p$.

Let $|G|_p = p^a$ and $|N|_p = p^b$, so $|P| = p^a$. Note that $P \cap N$ is a p-subgroup of N, and so

$$|P \cap N| \leqslant |N|_p = p^b.$$

By the Second Isomorphism Theorem, we have $PN/N \cong P/(P \cap N)$. Hence PN/N is a p-subgroup of G/N, and so

$$|PN/N| \leqslant |G/N|_p = \frac{|G|_p}{|N|_p} = \frac{p^a}{p^b}.$$

Therefore, we have

$$\frac{p^a}{p^b} \geqslant |PN/N| = |P/(P \cap N)| = \frac{|P|}{|P \cap N|} = \frac{p^a}{|P \cap N|} \geqslant \frac{p^a}{p^b}.$$

It follows that all of the inequalities above are actually equalities, hence $|PN/N| = |G/N|_p$, so $PN/N \in \mathrm{Syl}_p(G/N)$, and $|P \cap N| = |N|_p$, so $P \cap N \in \mathrm{Syl}_p(N)$. □

#IV. *If G is a group of order 168 and P is a Sylow 7-subgroup of G, then either P is a normal subgroup of G or else the normalizer of P is a maximal subgroup of G.*

Proposition: *Let G be a finite group of order m, p a prime dividing m, and P a Sylow p-subgroup of G. If there is exactly one integer $d > 1$ such that $d \mid m$ and $d \equiv 1 (\mathrm{mod}\, p)$, then either $P \trianglelefteq G$ or $N_G(P)$ is a maximal subgroup of G.*

Proof. We will prove the Proposition. The case $|G| = 168$ and $p = 7$ then follows, as $d = 8$ is the only integer satisfying $d > 1$, $d \mid 168$, and $d \equiv 1 (\mathrm{mod}\, 7)$.

Let n_p be the number of Sylow p-subgroups of G. Since $n_p \mid m$ and $n_p \equiv 1 (\mathrm{mod}\, p)$, either $n_p = 1$ or $n_p = d$, by hypothesis. If $n_p = 1$, then $P \trianglelefteq G$ and we are finished, so we now assume $n_p = d$, so $N_G(P)$ is a proper subgroup of G and $|G : N_G(P)| = d$.

Suppose $N_G(P) \leqslant H \leqslant G$. Then $P \leqslant H$, so P is a Sylow p-subgroup of H. Moreover,

$$N_H(P) = H \cap N_G(P) = N_G(P).$$

Hence if $n_p(H)$ is the number of Sylow p-subgroups of H, then

$$n_p(H) = |H : N_H(P)| = |H : N_G(P)|.$$

We also know that $n_p(H) \equiv 1 (\mathrm{mod}\, p)$ and $n_p(H)$ divides $|H|$, which divides $|G| = m$, so that $n_p(H) \mid m$. By hypothesis, either $n_p(H) = 1$, so that $H = N_G(P)$, or $n_p(H) = d$, so that $H = G$. Therefore, $N_G(P)$ is a maximal subgroup of G. □

#VII. *A group G of order $3^3 \cdot 5 \cdot 13$ must have a normal Sylow 13-subgroup or a normal Sylow 5-subgroup.*

Proof. If G has a normal Sylow 13-subgroup, then we are finished, so we assume a Sylow 13-subgroup is not normal, so $n_{13} \neq 1$. Since $n_{13} \mid 3^3 \cdot 5$ and $n_{13} \equiv 1 (\text{mod } 13)$, we have $n_{13} = 27 = 3^3$. Hence, if $T \in \mathrm{Syl}_{13}(G)$, then $|G : N_G(T)| = 3^3$, and so $|N_G(T)| = 5 \cdot 13$.

Since 5 and 13 are primes and $13 \not\equiv 1 (\text{mod } 5)$, we know that $N_G(T)$ is cyclic, hence abelian. (See Example 1, page 72 of notes.) Let F be a Sylow 5-subgroup of $N_G(T)$, so that $|F| = 5$ and so F is also a Sylow 5-subgroup of G. Since $N_G(T)$ is abelian, we have $F \trianglelefteq N_G(T)$, and in particular, T normalizes F, i.e., $T \leqslant N_G(F)$.

Now $n_5 \mid 3^3 \cdot 13$ and $n_5 \equiv 1 (\text{mod } 5)$, so $n_5 = 1$ or $n_5 = 3^3 \cdot 13$. If $n_5 = 3^3 \cdot 13$, then $|G : N_G(F)| = 3^3 \cdot 13$, so $|N_G(F)| = 5$, contradicting $T \leqslant N_G(F)$. Hence $n_5 = 1$ and $F \trianglelefteq G$. □

Note on §5.1 #4: In general, if $H \leqslant A \times B$, it does NOT follow that $H = A_1 \times B_1$ for $A_1 \leqslant A$, $B_1 \leqslant B$. For example, $\{(0,0),(1,1)\} \leqslant \mathbb{Z}_2 \times \mathbb{Z}_2$ is not a direct product of subgroups of $\mathbb{Z}_2$. However, **if** $H = A_1 \times B_1$, then

$$A_1 = \{a \in A \mid (a,b) \in H \text{ for some } b \in B\},$$

$$B_1 = \{b \in B \mid (a,b) \in H \text{ for some } a \in A\}.$$

In any case, H is a subgroup of $A_1 \times B_1$, but H may be a *proper* subgroup of $A_1 \times B_1$. In the example, we have $A_1 = \mathbb{Z}_2$ and $B_1 = \mathbb{Z}_2$, hence $\{(0,0),(1,1)\} < A_1 \times B_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

§5.1 #4: *Let A and B be finite groups and p a prime. If $H \in \mathrm{Syl}_p(A \times B)$, then $H = P \times Q$ for some $P \in \mathrm{Syl}_p(A)$ and $Q \in \mathrm{Syl}_p(B)$. Moreover, $n_p(A \times B) = n_p(A)n_p(B)$.*

Proof 1 [Easy Proof]. Let $P \in \mathrm{Syl}_p(A)$ and $Q \in \mathrm{Syl}_p(B)$. Then $P \times Q \leqslant A \times B$ and

$$|P \times Q| = |P| \cdot |Q| = |A|_p \cdot |B|_p = (|A| \cdot |B|)_p = |A \times B|_p,$$

so $P \times Q$ is a Sylow p-subgroup of $A \times B$.

Let $H \in \mathrm{Syl}_p(A \times B)$. By Sylow's Theorem, H is conjugate to $P \times Q$, so for some element (a,b) of $A \times B$, we have

$$H = (a,b)(P \times Q)(a,b)^{-1} = (a,b)(P \times Q)(a^{-1}, b^{-1}) = aPa^{-1} \times bQb^{-1},$$

and $aPa^{-1} \in \mathrm{Syl}_p(A)$ and $bQb^{-1} \in \mathrm{Syl}_p(B)$. Hence there is a bijection between $\mathrm{Syl}_p(A \times B)$ and the Cartesian product $\mathrm{Syl}_p(A) \times \mathrm{Syl}_p(B)$, and so $n_p(A \times B) = n_p(A)n_p(B)$. □

Proof 2 [Informative Proof]. As above, if $P \in \mathrm{Syl}_p(A)$, and $Q \in \mathrm{Syl}_p(B)$, then $P \times Q$ is a Sylow p-subgroup of $A \times B$.

Let $H \in \mathrm{Syl}_p(A \times B)$ and let A_1 and B_1 be defined as in the Note above. It is straightforward to show that $A_1 \leqslant A$ and $B_1 \leqslant B$ (exercise), and so $H \leqslant A_1 \times B_1 \leqslant A \times B$.

If $(a,b) \in H$, then $|(a,b)| = [|a|, |b|]$ is a power of p since H is a p-group. Hence $|a|$ and $|b|$ are both powers of p. It then follows that the order of each element of A_1 or B_1 is a power of p, so A_1 and B_1 are both p-groups. By Sylow's Theorem, $A_1 \leqslant P$ for some $P \in \mathrm{Syl}_p(A)$ and $B_1 \leqslant Q$ for some $Q \in \mathrm{Syl}_p(B)$. Hence

$$H \leqslant A_1 \times B_1 \leqslant P \times Q,$$

and since H and $P \times Q$ are both Sylow p-subgroups of $A \times B$, we have $|H| = |P \times Q|$. Hence $H = P \times Q$. It follows that $n_p(A \times B) = n_p(A)n_p(B)$ as before. □

§5.4 #11: *If $G = HK$, where H and K are characteristic subgroups of G with $H \cap K = \langle 1 \rangle$, then* $\mathrm{Aut}(G) \cong \mathrm{Aut}(H) \times \mathrm{Aut}(K)$.

Proof. Since H and K are characteristic subgroups of G, they are also normal. In particular, we have $G \cong H \times K$, so every $g \in G$ can be written uniquely in the form $g = hk$, $h \in H$, $k \in K$, and $hk = kh$ for all $h \in H$, $k \in K$. Also, if $\sigma \in \mathrm{Aut}(G)$, then $\sigma(H) = H$ and $\sigma(K) = K$. Hence the restrictions σ_H, σ_K of σ to H, K, respectively, are automorphisms.

Define $\rho : \mathrm{Aut}(G) \to \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ by $\rho(\sigma) = (\sigma_H, \sigma_K)$ for all $\sigma \in \mathrm{Aut}(G)$. We claim ρ is an isomorphism.

First, for $\sigma, \mu \in \mathrm{Aut}(G)$, we have

$$\begin{aligned} \rho(\sigma \circ \mu) &= ((\sigma \circ \mu)_H, (\sigma \circ \mu)_K) \\ &= (\sigma_H \circ \mu_H, \sigma_K \circ \mu_K) \\ &= (\sigma_H, \sigma_K) \circ (\mu_H, \mu_K) \\ &= \rho(\sigma) \circ \rho(\mu). \end{aligned}$$

Hence ρ is a homomorphism.

If $\sigma \in \ker \rho$, then $\rho(\sigma) = (\sigma_H, \sigma_K)$ is the identity element of $\mathrm{Aut}(H) \times \mathrm{Aut}(K)$. Hence σ_H is the identity map on H and σ_K is the identity map on K. Thus for $g = hk \in G$, with $h \in H$, $k \in K$, we have

$$\sigma(g) = \sigma(hk) = \sigma(h)\sigma(k) = hk = g,$$

and so σ is the identity map on G. Hence $\ker \rho$ is trivial and ρ is injective.

It remains to show that ρ is surjective. For $(\tau, \mu) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$, define $\varphi : G \to G$ by

$$\varphi(g) = \varphi(hk) = \tau(h)\mu(k)$$

for $g = hk \in G$, $h \in H$, $k \in K$. Since the expression $g = hk$ is unique, φ is well-defined. We must show that $\varphi \in \mathrm{Aut}(G)$ and $\rho(\varphi) = (\tau, \mu)$.

For $g = hk, g_1 = h_1 k_1 \in G$, we have

$$\begin{aligned} \varphi(gg_1) &= \varphi((hk)(h_1 h_1)) \\ &= \varphi((hh_1)(kk_1)) \text{ since } h_1 \in H, k \in K, \\ &= \tau(hh_1)\mu(kk_1) \\ &= [\tau(h)\tau(h_1)][\mu(k)\mu(k_1)] \text{ since } \tau, \mu \text{ are homomorphisms,} \\ &= [\tau(h)\mu(k)][\tau(h_1)\mu(k_1)] \text{ since } \tau(h_1) \in H, \mu(k) \in K, \\ &= \varphi(g)\varphi(g_1). \end{aligned}$$

Hence φ is a homomorphism.

If $g = hk \in \ker \varphi$, then $1_G = \varphi(g) = \tau(h)\mu(k)$, hence $\tau(h) = \mu(k)^{-1} \in H \cap K = \langle 1_G \rangle$. Therefore, $\tau(h) = \mu(k) = 1_G$, and since τ and μ are injective, this implies $h = k = 1_G$, and so $g = hk = 1_G$. Hence $\ker \varphi$ is trivial and φ is injective.

If $g = hk \in G$, then since τ and μ are surjective, there exist $h' \in H$, $k' \in K$ with $\tau(h') = h$ and $\mu(k') = k$. Then

$$\varphi(h'k') = \tau(h')\mu(k') = hk = g,$$

and so φ is surjective. Hence $\varphi \in \mathrm{Aut}(G)$.

Finally, if $h \in H$, then $\varphi(h) = \tau(h)\mu(1_G) = \tau(h)$ and if $k \in K$, then $\varphi(k) = \tau(1_G)\mu(k) = \mu(k)$. Hence $\varphi_H = \tau$ and $\varphi_K = \mu$, so that $\rho(\varphi) = (\varphi_H, \varphi_K) = (\tau, \mu)$. Therefore ρ is surjective and hence is an isomorphism. □