

PATTERNS, CONTINUED: VERIFYING FORMULAS

DIRECT VERIFICATION

Some explicit formulas can be verified, or proved, by arguing directly using known formulas or by referring to geometric patterns that are easy to verify. For example, recall that we verified that the sum of the n^{th} “up and down” sequence is n^2 and the sum of the first n odd counting numbers is also n^2 by considering the $n \times n$ square array of dots.

The following activities are designed to construct proofs of some of the formulas we have considered.

1. Recall the **Pool Tiling** problem: given a square pool, we wish to lay one-foot square tiles around the pool to form a walking border. We conjectured that the number of tiles required for a pool with side length n feet is $4n + 4$.

One way to verify this formula is as follows:

- a. Count the number of tiles on *each* side of the pool. (Count only those that share a side with the side of the pool, not corners.) Multiply this by 4 since there are four sides.
- b. Count the number of corner tiles required.
- c. Add the results of (a) and (b). This is the number of tiles required.

Can you describe another way to verify the formula?

2. Recall the **Polygon Triangulation** problem: given a polygon, choose a vertex and draw all possible diagonals starting at that vertex. By considering the way the pattern began, we conjectured that the number of triangles created in a polygon with n sides is $n - 2$.

One way to verify this formula is as follows:

- a. Determine the number of vertices in a polygon with n sides.
- b. Use (a) to determine the number of diagonals that can be drawn.
- c. Determine the relationship between the number of diagonals and the number of triangles.
- d. Use (b) and (c) to determine the number of triangles created.

Can you describe another way to verify the formula?

PATTERNS AND VERIFYING FORMULAS, continued

MATHEMATICAL INDUCTION

A technique called Mathematical Induction can be used to verify a formula or expression for the n^{th} term in a pattern. This method is particularly useful when it is easy to describe the pattern recursively; that is, whenever it is easy to describe the procedure for going from the n^{th} term to the $(n + 1)^{\text{th}}$ term in the pattern.

Suppose we have a conjecture for a formula for the n^{th} term in a pattern.

The **Principle of Mathematical Induction** says that if

1. the formula works for the case where $n = 1$, and
2. every time the formula works for one counting number, it also works for the next counting number,

then the formula works for *every* counting number.

Notice that Mathematical Induction does not help us to *find* the formula, but is only used to *verify* or *prove* a formula that we suspect to be true.

In practice, this technique is implemented as follows:

Step 1: Find a proposed formula for the n^{th} term in a pattern.

Step 2: Substitute 1 for n in the formula and verify that the result is the first term in the pattern.

Step 3: Find an expression for the $(k + 1)^{\text{th}}$ term in the pattern *assuming* the formula works for the k^{th} term, as follows:

- a. Substitute k for n in the formula to find an expression for the value of the k^{th} term in the pattern, according to the formula.
- b. Use the expression from Step 3(a) for the k^{th} term and apply the recursive procedure to find the expression for the $(k + 1)^{\text{th}}$ term.

Step 4: Find an expression for the $(k + 1)^{\text{th}}$ term in the pattern *according to the formula* by substituting $k + 1$ for n in the formula.

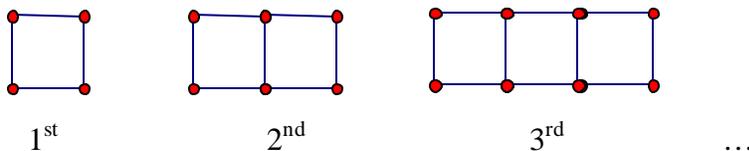
Step 5: Verify that the expressions for the $(k + 1)^{\text{th}}$ term found in Steps 3 and 4 are equal.

Step 2 is called the **base step**; it verifies that condition (1) in the Principle of Mathematical Induction holds. The assumption in Step 3 that the formula works for the k^{th} term is called the **inductive hypothesis**. Steps 3 – 5 together are called the **induction step**; they verify that condition (2) in the Principle of Mathematical Induction holds. Thus, according to the Principle of Mathematical Induction, Steps 2 – 5 guarantee that the formula found in Step 1 gives the n^{th} term in the pattern for every counting number n .

PATTERNS AND VERIFYING FORMULAS, continued

Problems That Use Induction

1. Recall the **Square Trains** problem: find the number of “external sides” in the square train made up of n squares, as in the picture.



Step 1: We conjectured that the number of external sides on a train with n squares is $2n + 2$.

The following steps establish this formula for all counting numbers n :

Step 2: Verify that the formula works when $n = 1$:

- When $n = 1$, there is 1 square. How many external sides are there?
- What does the formula (that is, $2n + 2$) give for the number of external sides when $n = 1$?
- Are your answers to (a) and (b) the same? If so, then the formula works when $n = 1$.

Step 3: Now determine the number of external sides in a train of $k + 1$ squares, *assuming* the formula works for a train of k squares:

- How many external sides does the *formula* say a train of k squares has? (Substitute k for n in the formula.)
- Assuming the expression in Step 3(a), determine how many external sides a train of $k + 1$ squares has:
 - Add one square to the train to obtain a train with $k + 1$ squares. How many external sides does this *add* to the train?
 - Assuming the results of Step 3(a) and 3(b)(i), what is the total number of external sides in a train with $k + 1$ squares?

Step 4: How many external sides does the *formula* say a train with $k + 1$ squares should have? (Substitute $k + 1$ for n in the formula.)

Step 5: Are your answers for the number of external sides in a train of $k + 1$ sides from Step 3 and Step 4 the same?

The Principle of Mathematical Induction says that if the answers to Step 1(c) and Step 5 are both *yes*, then the formula must work for every counting number. That is, the number of external sides of a square train with n squares is $2n + 2$.