Properties of 3×3 Magic Squares

Notation: We will denote the entries in a 3×3 magic square as follows:

C_1	S_1	C_2
S_4	C	S_2
C_4	S_3	C_3

We will denote by M the magic number of the square; that is, M is the sum of the 3 entries in each row, each column, and each diagonal.

Theorem 1. If a 3×3 magic square has magic number M, then the sum of the 9 entries in the square is $3M$. That is, in the notation above,

$$C + C_1 + C_2 + C_3 + C_4 + S_1 + S_2 + S_3 + S_4 = 3M.$$

Proof. Since the sum of the entries in each row of the magic square is M, we know that

$$\begin{aligned} C_1 + S_1 + C_2 &= M \\ S_4 + C + S_2 &= M \\ C_4 + S_3 + C_3 &= M. \end{aligned}$$

Therefore, rearranging and regrouping using the associative and i commutative laws of addition, we have

$$\begin{aligned} C + C_1 + C_2 + C_3 + C_4 + S_1 + S_2 + S_3 + S_4 &= \\ (C_1 + S_1 + C_2) + (S_4 + C + S_2) + (C_4 + S_3 + C_3) &= \\ M \quad + \quad M \quad + \quad M &= 3M. \end{aligned}$$

Hence,

$$C + C_1 + C_2 + C_3 + C_4 + S_1 + S_2 + S_3 + S_4 = 3M,$$

as claimed. □

Theorem 2. The magic number of a 3×3 magic square is three times the number in the center square. That is, in the notation above, $M = 3C$, or equivalently, $C = \frac{1}{3}M$.

Proof. Since the sum of the entries in each row, column, or diagonal is the magic number M, we have

$$\begin{aligned} C_1 + C + C_3 &= M \\ C_2 + C + C_4 &= M \\ S_1 + C + S_3 &= M \\ S_2 + C + S_4 &= M. \end{aligned}$$

Therefore,

$$\begin{aligned} (C_1 + C + C_3) + (C_2 + C + C_4) + (S_1 + C + S_3) + (S_2 + C + S_4) &= \\ M \quad + \quad M \quad + \quad M \quad + \quad M &= 4M. \end{aligned}$$

Rearranging and regrouping using the associative and commutative laws of addition, we have

$$(C + C + C) + (C + C_1 + C_2 + C_3 + C_4 + S_1 + S_2 + S_3 + S_4) = 4M.$$

By Theorem 1, we can substitute $3M$ for $C+C_1+C_2+C_3+C_4+S_1+S_2+S_3+S_4$ to obtain

$$(C + C + C) + 3M = 4M,$$

and, subtracting $3M$ from both sides of this equation, we have

$$C + C + C = M.$$

Hence $3C = M$, or equivalently, $C = \frac{1}{3}M$ as claimed. $\square$

Theorem 3. The number in the center square of a 3×3 magic square is the median of the 9 entries in the square. That is, in the notation above, C is the median of the numbers C, C_1, C_2, C_3, C_4, S_1, S_2, S_3, and S_4.

Proof. We know that the sum of the entries in each row, column, or diagonal is the magic number M. By Theorem 2, $M = 3C$, so substituting $3C$ for M shows that the sum of the entries in each row, column, or diagonal is $3C$. Hence we have

$$\begin{aligned} C_1 + C + C_3 &= 3C \\ C_2 + C + C_4 &= 3C \\ S_1 + C + S_3 &= 3C \\ S_2 + C + S_4 &= 3C. \end{aligned}$$

Subtracting C from both sides of each equation yields

$$\begin{aligned} C_1 + C_3 &= 2C \\ C_2 + C_4 &= 2C \\ S_1 + S_3 &= 2C \\ S_2 + S_4 &= 2C. \end{aligned}$$

Now if *both* of C_1 and C_3 are greater than C, then $C_1 + C_3$ is greater than $C + C$, and if *both* of C_1 and C_3 are less than C, then $C_1 + C_3$ is less than $C + C$. But the first equation above says that $C_1 + C_3$ is *equal* to $2C = C + C$. Therefore, either

1. one of C_1 or C_3 is *greater* than C and the other is *less* than C, or
2. both C_1 and C_3 are *equal* to C.

By the same argument, this holds for each of the pairs of numbers

$$\{C_1, C_3\},\ \{C_2, C_4\},\ \{S_1, S_3\},\ \text{and}\ \{S_2, S_4\}.$$

Therefore, exactly half of the numbers

$$C_1,\ C_2,\ C_3,\ C_4,\ S_1,\ S_2,\ S_3,\ \text{and}\ S_4$$

are greater than or equal to C and the other half are less than or equal to C. Thus, by the definition of median, C is the median of the numbers

$$C,\ C_1,\ C_2,\ C_3,\ C_4,\ S_1,\ S_2,\ S_3,\ \text{and}\ S_4$$

as claimed. ☐