Primary Decomposition (§10.5)

Our goal now is to show that if $\Delta(t)$ splits into linear factors, we get a decomposition as in the Theorem above, where the $W_i$ are the generalized eigenspaces.

Lemma 1: Let $T$ be a linear operator on $V$ and $f(t)$ a polynomial with $f(T)=0$. If $f(t)=g(t)h(t)$ with $g(t), h(t)$ relatively prime, then $U=\ker g(T)$, $W=\ker h(T)$ are $T$-invariant subspaces and $V=U\oplus W$.

Proof: Since $U, W$ are kernels of polynomials in $T$, they are $T$-invariant by previous results.

As $g(t), h(t)$ are relatively prime, there are polynomials $a(t), b(t)$ such that $a(t)g(t)+b(t)h(t)=1$ [Polynomials, #6]. Hence $a(T)\circ g(T)+b(T)\circ h(T)=\mathcal{I}$, the identity map.

$V=U+W$: By hypothesis, $f(T)=g(T)\circ h(T)=0$ (zero map on $V$) and we know all polynomials in $T$ commute [Polynomials, #10].

Let $\bar{v}\in V$. Then $\bar{v}=\mathcal{I}(\bar{v})=a(T)\circ g(T)(\bar{v})+b(T)\circ h(T)(\bar{v})$.
Let $\bar{w}=a(T)\circ g(T)(\bar{v})$ and $\bar{u}=b(T)\circ h(T)(\bar{v})$, so $\bar{v}=\bar{u}+\bar{w}$.
We have $g(T)(\bar{u})=g(T)\circ b(T)\circ h(T)(\bar{v})=b(T)(g(T)\circ h(T)(\bar{v}))=b(T)(\bar{0})=\bar{0}$, and so $\bar{u}\in\ker g(T)=U$. Similarly,
$h(T)(\bar{w})=h(T)\circ a(T)\circ g(T)(\bar{v})=a(T)(g(T)\circ h(T)(\bar{v}))=a(T)(\bar{0})=\bar{0}$, and so $\bar{w}\in\ker h(T)=W$. Hence $V=U+W$.

$U\cap W=\{\bar{0}\}$ Suppose $\bar{v}\in U\cap W=\ker g(T)\cap\ker h(T)$. We have $g(T)(\bar{v})=\bar{0}$ and $h(T)(\bar{v})=\bar{0}$. Therefore,

$$\begin{aligned}\bar{v}=\mathcal{I}(\bar{v})&=a(T)\circ g(T)(\bar{v})+b(T)\circ h(T)(\bar{v})\\&=a(T)(g(T)(\bar{v}))+b(T)(h(T)(\bar{v}))\\&=a(T)(\bar{0})+b(T)(\bar{0})=\bar{0}+\bar{0}=\bar{0}.\end{aligned}$$

Hence if $\bar{v}\in U\cap W$ then $\bar{v}=\bar{0}$, and so $U\cap W=\{\bar{0}\}$.

Therefore, $V=U\oplus W$. [See Problem 10.9 for a different proof.] $\square$