For the next few results, we assume the following notation:

$$(*)\quad \left[\begin{array}{l} T \text{ is a linear operator on } V \\ m(t) = m_T(t) = f_1(t)^{l_1} f_2(t)^{l_2} \cdots f_r(t)^{l_r}\text{, the } f_i(t) \text{ are} \\ \qquad \text{distinct, monic, irreducible polynomials, } l_i \geq 1 \text{ each } i \\ W_i = \ker f_i(T)^{l_i} \text{ for each } i\text{, so } V = W_1 \oplus W_2 \oplus \cdots \oplus W_r \\ T_i = \text{restriction of } T \text{ to } W_i \\ m_i(t) = f_i(t)^{l_i}\text{, the minimal polynomial of } T_i \\ \Delta(t) = \Delta_T(t) = f_1(t)^{n_1} f_2(t)^{n_2} \cdots f_r(t)^{n_r}\text{, } n_i \geq l_i \text{ for each } i \\ \Delta_i(t) = \text{characteristic polynomial of } T_i \text{ for each } i. \end{array}\right.$$

Note: We know $\Delta(t)$ is of the given form by basic properties of minimal and characteristic polynomials

We next show that $\Delta_i(t) = f_i(t)^{n_i}$ for each i.

Corollary: If $(*)$ holds, then $\Delta_i(t) = f_i(t)^{n_i}$ and $\dim W_i = \deg f_i(t)^{n_i}$.

Proof: Since $V = W_1 \oplus \cdots \oplus W_r$, we know $\Delta(t) = \Delta_1(t)\Delta_2(t)\cdots\Delta_r(t)$. By the Primary Decomposition Theorem $m_i(t) = f_i(t)^{l_i}$, and so $\Delta_i(t) = f_i(t)^{k_i}$ for some $k_i \geq l_i$ (irreducible factors of Δ_i, m_i equal).

We have $\Delta(t) = f_1(t)^{n_1} f_2(t)^{n_2} \cdots f_r(t)^{n_r} = f_1(t)^{k_1} f_2(t)^{k_2} \cdots f_r(t)^{k_r}$ with the $f_i(t)$ distinct and irreducible. Hence by unique factorization [Polynomials, #9], this implies $n_i = k_i$ for all i, and so $\Delta_i(t) = f_i(t)^{n_i}$ for all i.

Finally, in general if $F: U \to U$ is linear, then $\dim U = \deg \Delta_F(t)$. In our case, this says $\dim W_i = \deg \Delta_i(t) = \deg f_i(t)^{n_i}$. $\square$

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