Lec 35, 4/24/09

Lemma: Assume notation (*). If $j \neq i$ and $n \geq 1$, then the restriction $(f_i(T)^n)_{W_j}$ of $f_i(T)^n$ to W_j is nonsingular (one to one).

Proof: We need to show that $\ker (f_i(T)^n)_{W_j} = \{\bar{0}\}$, that is, if $\bar{w} \in W_j$ and $f_i(T)^n(\bar{w}) = \bar{0}$, then $\bar{w} = \bar{0}$.

Suppose to the contrary that $\bar{w} \in W_j$, $\underline{\bar{w} \neq \bar{0}}$, and $f_i(T)^n(\bar{w}) = \bar{0}$. Let d be the smallest positive integer such that $f_i(T)^d(\bar{w}) = \bar{0}$ (thus $1 \leq d \leq n$). Now let $\bar{y} = f_i(T)^{d-1}(\bar{w})$, so that $\underline{\bar{y} \neq \bar{0}}$.

Since W_j is T-invariant, W_j is invariant under any polynomial in T. Hence W_j is $f_i(T)^{d-1}$ invariant, and so $\underline{\bar{y} \in W_j}$.

We also have $\bar{0} = f_i(T)^d(\bar{w}) = f_i(T)(f_i(T)^{d-1}(\bar{w})) = f_i(T)(\bar{y})$.

Hence $\bar{y} \in \ker f_i(T) \subseteq \ker f_i(T)^{\ell_i} = W_i$, and so $\underline{\bar{y} \in W_i}$.

By the Primary Decomposition Theorem, $W_i \cap W_j = \{\bar{0}\}$, hence $\underline{\bar{y} = \bar{0}}$, a contradiction. Hence $\bar{w} = \bar{0}$ as claimed. ∎

Theorem: Assume notation (*). If $n \geq 1$, then $\ker f_i(T)^n \subseteq \ker f_i(T)^{\ell_i} = W_i$ for all i. In particular, $\ker f_i(T)^n = \ker f_i(T)^{\ell_i} = W_i$ for all $n \geq \ell_i$.

Proof: We have $V = W_1 \oplus \cdots \oplus W_r$ and each W_j is T-invariant, hence also $f_i(T)^n$-invariant. Let $\bar{v} \in \ker f_i(T)^n$, so that $f_i(T)^n(\bar{v}) = \bar{0}$, we must show $\bar{v} \in W_i$.

We can write $\bar{v} = \bar{w}_1 + \cdots + \bar{w}_i + \cdots + \bar{w}_r$, where $\bar{w}_j \in W_j$ all j. We have $\bar{0} = f_i(T)^n(\bar{v}) = f_i(T)^n(\bar{w}_1) + \cdots + f_i(T)^n(\bar{w}_i) + \cdots + f_i(T)^n(\bar{w}_r)$, and $f_i(T)^n(\bar{w}_j)$ is in W_j for all j. Thus by uniqueness of expression in a direct sum, we have $f_i(T)^n(\bar{w}_j) = \bar{0}$ for all j.

By the Lemma, $f_i(T)^n$ is nonsingular on W_j for $j \neq i$. Thus for $j \neq i$, we have $\bar{w}_j = \bar{0}$. Therefore $\bar{v} = \bar{w}_i$ and so $\bar{v} \in W_i$ as claimed.

[Continued ⟶]