[Proof, cont.] If, in particular, $n \geq l_i$, then we also have $\ker f_i(T)^{l_i} \subseteq \ker f_i(T)^n$. [Exercise. See HW #11, Problem #4.] Hence $\ker f_i(T)^n = \ker f_i(T)^{l_i}$ for all $n \geq l_i$. $\square$

Recall that if λ is an eigenvalue of T with algebraic multiplicity n, then $K_\lambda = \ker(T-\lambda I)^n$ is the generalized eigenspace of T belonging to λ.

Corollary: If T is a linear operator on V with $\Delta_T(t) = (t-\lambda)^n g(t)$ and $m_T(t) = (t-\lambda)^l h(t)$, where $t-\lambda$ is not a factor of $g(t)$ or $h(t)$, then $K_\lambda = \ker(T-\lambda I)^l$ and $\ker(T-\lambda I)^m = K_\lambda$ for all $m \geq l$.

Proof: Let $f_1(t)^{l_1} = (t-\lambda)^l$ in the notation and theorem above. $\square$

We now specialize notation (*) to the case where $\Delta_T(t)$ splits:

$$(**)\quad \begin{cases} T \text{ is a linear operator on } V \\ m(t) = m_T(t) = (t-\lambda_1)^{l_1}(t-\lambda_2)^{l_2}\cdots(t-\lambda_r)^{l_r}, \ \lambda_i \text{ distinct eigenvalues.} \\ \Delta(t) = \Delta_T(t) = (t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2}\cdots(t-\lambda_r)^{n_r}, \quad 1 \leq l_i \leq n_i. \\ E_i = E_{\lambda_i} = \ker(T-\lambda_i I) \text{ is the } \lambda_i\text{-eigenspace.} \\ K_i = K_{\lambda_i} = \ker(T-\lambda_i I)^{n_i} = \ker(T-\lambda_i I)^{l_i} \text{ is the generalized } \lambda_i\text{-eigenspace.} \\ T_i = T_{K_{\lambda_i}} \text{ is the restriction of } T \text{ to } K_i. \\ \Delta_i(t) = \text{characteristic polynomial of } T_i \\ m_i(t) = \text{minimal polynomial of } T_i. \end{cases}$$

Specializing the results above to this case we obtain:

Theorem: Assuming notation (**), the following hold:

(i) $V = K_1 \oplus K_2 \oplus \cdots \oplus K_r$, each K_i is T-invariant

(ii) $m_i(t) = (t-\lambda_i)^{l_i}$

(iii) $\Delta_i(t) = (t-\lambda_i)^{n_i}$

(iv) For $j \neq i$, the restriction of $T-\lambda_i I$ to K_j is nonsingular

(v) $\dim K_i = n_i =$ algebraic multiplicity of λ_i.