<u>Corollary</u>: Assume notation (**). Construct a basis B_i for K_i by finding a basis for $E_i = \ker(T-\lambda_i I)$, extending to $\ker(T-\lambda_i I)^2$, etc., up to a basis for $K_i = \ker(T-\lambda_i I)^{l_i}$.

If $B = B_1 \cup B_2 \cup \cdots \cup B_r$, then B is a basis for V and

$$[T]_B = \begin{bmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_r \end{bmatrix}$$ is block diagonal, where

$$A_i = [T_i]_{B_i} = \begin{bmatrix} \lambda_i & & & * \\ & \lambda_i & & \\ & & \ddots & \\ O & & & \lambda_i \end{bmatrix}$$ is an upper triangular $n_i \times n_i$ matrix with λ_i on the diagonal.

<u>Corollary</u>: A linear operator T on V is diagonalizable if and only if $m_T(t)$ factors as a product of <u>distinct</u> <u>linear</u> factors.

<u>Proof</u>: First note that if T is diagonalizable, then $\Delta_T(t)$ and hence $m_T(t)$ factors over K into linear factors. Thus we may assume notation (**) holds. Now T is diagonalizable if and only if a basis of eigenvectors exists, hence if and only if $E_i = K_i$ for all i.

$\boxed{\Leftarrow}$ If $m_T(t)$ is a product of <u>distinct</u> linear factors, then $l_i = 1$ for all i. Hence $E_i = K_i$ and T is diagonalizable.

$\boxed{\Rightarrow}$ If T is diagonalizable, then $K_i = E_i$ for all i, and so $V = E_1 \oplus \cdots \oplus E_r$. If $\bar{v} \in V$, then $\bar{v} = \bar{w}_1 + \cdots + \bar{w}_r$, where $w_i \in E_i$, that is, $\bar{w}_i$ is an eigenvector for λ_i, and so $(T-\lambda_i I)(\bar{w}_i) = \bar{0}$ for all i. Hence

$$(T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_r I)(\bar{w}_i) = \bar{0} \text{ for all } i,$$

and so $(T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_r I)$ is the zero operator.

End Lec 35 | Therefore $m_T(t)$ divides, hence equals, $(t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_r)$. ∎