Lec 36, 4/27/09 

## Jordan Canonical Form

Defn: A Jordan block of size $n$ for the eigenvalue $\lambda$ is the $n \times n$ matrix with all diagonal entries $\lambda$, all 1 on the super diagonal and 0 elsewhere:

$$\begin{bmatrix} \lambda & 1 & & O \\ & \lambda & 1 & \\ & & \ddots & 1 \\ O & & & \lambda \end{bmatrix}$$

For example, Jordan blocks for $\lambda = 2$, of size 1, 2, 3, 4 are

$$[2], \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

However, the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is not a Jordan **block**.

We have seen previously that if $T$ is a linear operator on $V$ with $\Delta_T(t) = (t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2}\cdots(t-\lambda_r)^{n_r}$, $\lambda_i$ distinct eigenvalues, then $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_r}$ and we can choose bases $S_i$ for $K_{\lambda_i}$, and $S = S_1 \cup S_2 \cup \cdots \cup S_r$ is a basis for $V$ such that

$$[T]_S = \begin{bmatrix} A_1 & & O \\ & A_2 & \\ & & \ddots \\ O & & & A_r \end{bmatrix} \text{ is block diagonal, } A_i = [T_i]_{S_i} = \begin{bmatrix} \lambda_i & & * \\ & \lambda_i & \\ O & & \ddots \\ & & & \lambda_i \end{bmatrix}$$

This basis can be further refined so that each $A_i$ is also block diagonal, where each block of $A_i$ is a Jordan block for the eigenvalue $\lambda_i$.

Defn A matrix $J$ is in Jordan canonical form if $J$ is block diagonal and each block is a Jordan block for some eigenvalue of $J$.