(EXAMPLES, cont.)

⑥ Subspaces associated with a given matrix.

Let A be a fixed $m \times n$ matrix with real entries.
The following are subspaces of the indicated spaces.

$W_1 = \mathcal{N}(A) =$ nullspace of A $= \{\bar{x} \in \mathbb{R}^n_{col} \mid A\bar{x} = \bar{0}\} \subseteq \mathbb{R}^n_{col}$

$W_2 =$ left nullspace of A $= \{\bar{y} \in \mathbb{R}^m_{row} \mid \bar{y}A = \bar{0}\} \subseteq \mathbb{R}^m_{row}$

$W_3 = \mathcal{C}(A) =$ column space of A $= \{\bar{z} \in \mathbb{R}^m_{col} \mid \bar{z} = A\bar{x}$ for some $\bar{x} \in \mathbb{R}^n_{col}\} \subseteq \mathbb{R}^m_{col}$
= set of all linear combinations of columns of A.

$W_4 = \mathcal{R}(A) =$ rowspace of A $= \{\bar{w} \in \mathbb{R}^n_{row} \mid \bar{w} = \bar{x}A$ for some $\bar{x} \in \mathbb{R}^m_{row}\} \subseteq \mathbb{R}^n_{row}$
= set of all linear combinations of rows of A.

Proof: W_1 is proved in the text [Theorem 4.4].
W_4 is left as an exercise [similar to W_3].

W_2: (i) Since $\bar{0}_m A = \bar{0}_n$, we have $\bar{0}_m \in W_2$ and $W_2 \neq \emptyset$.
(ii) if $\bar{y}_1, \bar{y}_2 \in W_2$, so that $\bar{y}_1 A = \bar{0}$, $\bar{y}_2 A = \bar{0}$, then
$(\bar{y}_1 + \bar{y}_2)A = \bar{y}_1 A + \bar{y}_2 A$ by theorem 2.2,
$= \bar{0} + \bar{0} = \bar{0}$.
Hence $\bar{y}_1 + \bar{y}_2 \in W_2$ and W_2 is closed under vector addition.
(iii) If $\bar{y} \in W_2$, so that $\bar{y}A = \bar{0}$, and $\alpha \in \mathbb{R}$, then
$(\alpha\bar{y})A = \alpha(\bar{y}A)$ by theorem 2.2,
$= \alpha\bar{0} = \bar{0}$.
Hence $\alpha\bar{y} \in W_2$ and W_2 is closed under scalar multiplication.
Since (i), (ii), (iii) hold, W_2 is a subspace of $\mathbb{R}^m_{row}$.

W_3: (i) Since $\bar{0}_m = A\bar{0}_n$, we have $\bar{0}_m \in W_3$ and $W_3 \neq \emptyset$
(ii) If $\bar{z}_1, \bar{z}_2 \in W_3$, so there are $\bar{x}_1, \bar{x}_2 \in \mathbb{R}^n_{col}$ with $\bar{z}_1 = A\bar{x}_1$, $\bar{z}_2 = A\bar{x}_2$,
we have $\bar{z}_1 + \bar{z}_2 = A\bar{x}_1 + A\bar{x}_2 = A(\bar{x}_1 + \bar{x}_2)$, $\bar{x}_1 + \bar{x}_2 \in \mathbb{R}^n_{col}$. Hence
$\bar{z}_1 + \bar{z}_2 \in W_3$ and W_3 is closed under vector addition.
(iii) If $\bar{z} \in W_3$, so there is $\bar{x} \in \mathbb{R}^n_{col}$ with $\bar{z} = A\bar{x}$, and $\alpha \in \mathbb{R}$, then
$\alpha\bar{z} = \alpha(A\bar{x}) = A(\alpha\bar{x})$ and $\alpha\bar{x} \in \mathbb{R}^n_{col}$. Hence $\alpha\bar{z} \in W_3$
and so W_3 is closed under scalar multiplication.
Since (i), (ii), (iii) hold, W_3 is a subspace of $\mathbb{R}^m_{col}$. □