[Notes on Proof, continued]

For statement (2), observe that $m_T(T)=0$ and the multiplicity of $t-\lambda$ as a factor of $m_T(t)$ is the smallest power needed to "kill" all of the λ-blocks. Since $(J_\lambda-\lambda I)^m=0$ but $(J_\lambda-\lambda I)^{m-1}\neq 0$ for a λ-block J_λ of size m, this power is the size of the largest λ-block.

For statement (3), if A is the Jordan Form of T, then $A-\lambda I$ has one zero column for each λ-block (the first column in the block) and all other columns are linearly independent. Hence the number of blocks is equal to the number of zero columns of $A-\lambda I$, which is equal to $\dim \mathcal{N}(A-\lambda I) = \dim E_\lambda$.

For Statement (4), see text, Problems 10.16, 10.18. □

Remarks:

1. In Statement (4), $(T-\lambda \mathbb{1})^0$ should be interpreted as the identity operator, so $\text{nullity}(T-\lambda\mathbb{1})^0=0$, $\text{rank}(T-\lambda\mathbb{1})^0=\dim V$.

2. We have seen that $\ker(T-\lambda_i\mathbb{1})^{l_i}=\ker(T-\lambda_i\mathbb{1})^{l_i+1}=\dots$. Hence finding $\dim\ker(T-\lambda_i\mathbb{1})^k$ for $k=1,2,\dots,l_i$ and using (4) is always sufficient to find the Jordan Form. This, however, may be a lot of work and is necessary only if the Jordan Form is not uniquely determined by (1)-(3).

In fact, we have [exercises]:

- If every eigenvalue of T has algebraic multiplicity at most 3, then $\Delta_T(t)$, $m_T(t)$ completely determine Jordan Form.

- If every eigenvalue of T has algebraic multiplicity at most 6, then $\Delta_T(t)$, $m_T(t)$, and $\dim E_\lambda$ for each λ determine the Jordan Form.

End Lec 37