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[EXAMPLE 5, cont.]

(d) For the JCF of A found in (c), determine $m_i = \dim \mathcal{N}(A-2I)^i$ for all i.

Observe that if J is a Jordan block for λ of size m, then

$$J-\lambda I = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}.$$ There is one zero column and the remaining columns have pivots, hence $\dim \mathcal{N}(J-\lambda I)^1 = 1$. With each successive power, the 1's shift up one diagonal, increasing the number of zero columns by one. Hence $\dim \mathcal{N}(J-\lambda I)^k = k$ for $0 \le k \le m$. Since $(J-\lambda I)^m = 0$, $\dim \mathcal{N}(J-\lambda I)^m = m$ for all $k \geq m$.

In this example, we have

$$A = \left[\begin{array}{cccc|ccc|c} 2 & 1 & 0 & 0 & & & & \\ 0 & 2 & 1 & 0 & & & & \\ 0 & 0 & 2 & 1 & & & & \\ 0 & 0 & 0 & 2 & & & & \\ \hline & & & & 2 & 1 & 0 & \\ & & & & 0 & 2 & 1 & \\ & & & & 0 & 0 & 2 & \\ \hline & & & & & & & 2 \end{array}\right], \quad A-2I = \left[\begin{array}{cccc|ccc|c} 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \hline & & & & 0 & 1 & 0 & \\ & & & & 0 & 0 & 1 & \\ & & & & 0 & 0 & 0 & \\ \hline & & & & & & & 0 \end{array}\right]$$

Hence

$m_0 = \dim \mathcal{N}(A-2I)^0 = \dim \mathcal{N}(I) = 0$

$m_1 = \dim \mathcal{N}(A-2I) = \dim E_2 = 3$ (= # of blocks, as expected).

For each successive power, one zero column is added for each block that is not already 0, and so

$m_2 = \dim \mathcal{N}(A-2I)^2 = 5$

$m_3 = \dim \mathcal{N}(A-2I)^3 = 7$

$m_4 = \dim \mathcal{N}(A-2I)^4 = \dim \mathcal{N}(0) = 8$, and $m_i = 8$ for all $i \geq 4$.

If b_k is the number of blocks of size k, we have from the theorem:

$b_1 = -m_0 + 2m_1 - m_2 = 0 + 2(3) - 5 = 1$, $\quad b_3 = -m_2 + 2m_3 - m_4 = -5 + 2(7) - 8 = 1$

$b_2 = -m_1 + 2m_2 - m_3 = -3 + 2(5) - 7 = 0$, $\quad b_4 = -m_3 + 2m_4 - m_5 = -7 + 2(8) - 8 = 1$

(and of course $b_k = -m_{k-1} + 2m_k - m_{k-1} = -8 + 2(8) - 8 = 0$ for $k \geq 5$).