(116) [EXAMPLES, CONT]

⑥ Find the Jordan Canonical Form of $A = \begin{bmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{bmatrix}$.

$$\Delta_A(t) = \begin{vmatrix} t & 1 & 1 \\ 3 & t+1 & 2 \\ -7 & -5 & t-6 \end{vmatrix} = t(t+1)(t-6) - 15 - 14 + 7(t+1) + 10t - 3(t-6)$$

$$= t^3 - 5t^2 - 6t - 29 + 7t + 7 + 10t - 3t + 18$$

$$= t^3 - 5t^2 + 8t - 4 = (t-1)(t-2)^2.$$

Hence the JCF is $\left[\begin{array}{cc|c} 2 & 1 & \\ & 2 & \\ \hline & & 1 \end{array}\right]$ or $\left[\begin{array}{c|c|c} 2 & & \\ \hline & 2 & \\ \hline & & 1 \end{array}\right]$.

We need to determine if there is one block of size 2 or two blocks of size 1 for $\lambda = 2$. This can be determined by finding $m_A(t)$ (gives largest block size) or by finding $\dim E_2 = \dim \mathcal{N}(A - 2I)$ (gives number of blocks).

Method 1: $m_A(t)$ is either $(t-1)(t-2)$ or $(t-1)(t-2)^2$.

$$(A-I)(A-2I) = \begin{bmatrix} -1 & -1 & -1 \\ -3 & -2 & -2 \\ 7 & 5 & 5 \end{bmatrix} \begin{bmatrix} -2 & -1 & -1 \\ -3 & -3 & -2 \\ 7 & 5 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ -2 & -1 & -1 \\ 6 & 3 & 3 \end{bmatrix} \neq 0$$

Hence $m_A(t) = (t-1)(t-2)^2$ and the JCF is $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$.

Method 2: We have $A - 2I = \begin{bmatrix} -2 & -1 & -1 \\ -3 & -3 & -2 \\ 7 & 5 & 4 \end{bmatrix}$; and no column is a multiple of another, so the rank of $A - 2I$ is at least 2. Hence $\dim \mathcal{N}(A - 2I) \leq 1$, and there is at most one Jordan block for $\lambda = 2$. (Since $\lambda = 2$ is an eigenvalue, there is at least one also.) Again, the JCF is $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$.