⑦ Find the Jordan Canonical Form of
$$B = \begin{bmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{bmatrix},$$
with $\Delta_B(t) = (t-2)^2(t-4)^2$.

This will be determined if we find $m_B(t)$ (so largest block size) or the dimension of each eigenspace (number of blocks).

Method 1: $m_A(t)$ is one of $(t-2)(t-4)$, $(t-2)^2(t-4)$, $(t-2)(t-4)^2$ or $(t-2)^2(t-4)^2$.

We have
$$B-2I = \begin{bmatrix} 0 & -4 & 2 & 2 \\ -2 & -2 & 1 & 3 \\ -2 & -2 & 1 & 3 \\ -2 & -6 & 3 & 5 \end{bmatrix} \text{ and } B-4I = \begin{bmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{bmatrix}, \text{ and so}$$

$$(B-2I)(B-4I) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 2 \\ 0 & -4 & 2 & 2 \\ 0 & -4 & 2 & 2 \end{bmatrix} \neq 0, \text{ hence } m_B(t) \neq (t-2)(t-4).$$

$$(B-2I)^2(B-4I) = (B-2I)[(B-2I)(B-4I)] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -8 & 4 & 4 \\ 0 & -8 & 4 & 4 \\ 0 & -8 & 4 & 4 \end{bmatrix} \neq 0,$$

hence $m_B(t) \neq (t-2)^2(t-4)$.

$(B-2I)(B-4I)^2 = [(B-2I)(B-4I)](B-4I) = 0$ (check), hence $m(t) = (t-2)(t-4)^2$ and so there are two blocks of size 1 for $\lambda = 2$ and one block of size 2 for $\lambda = 4$.

Method 2: Find $\dim E_2$ and $\dim E_4$ to determine the number of blocks.

$$B-2I \longrightarrow \begin{bmatrix} -2 & -2 & 1 & 3 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$\dim \mathcal{N}(B-2I) = 2$, so two blocks for $\lambda = 2$, hence must be of size 1.

$$B-4I \longrightarrow \begin{bmatrix} -2 & -4 & 2 & 2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$\dim \mathcal{N}(B-4I) = 1$, so one block for $\lambda = 4$, hence must be of size 2.