Linear Combinations and Spanning Sets (§§ 4.4, 4.6)

In the following, V is a vectorspace over the field K

Defn: A linear combination of the vectors $\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n$ is a vector of the form $\bar{v} = \alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \cdots + \alpha_n\bar{v}_n$ for some $\alpha_1, \alpha_2, \cdots, \alpha_n$ in K.

Defn: Let S be a subset of V. The span of S, denoted $sp(S)$, is the set of all linear combinations of vectors in S. If $S = \emptyset$, we define $sp(S) = \{\bar{0}\}$.

Remark: If $\bar{v} \in S$, then $1\bar{v} = \bar{v}$ is a linear combination of vectors in S, so $\bar{v} \in sp(S)$. Hence $S \subseteq sp(S)$.

Theorem If S is a subset of V, then $sp(S)$ is a subspace of V containing S.

Proof: By the definition and remark, $sp(S)$ is non-empty and contains S. Any sum or scalar multiple of linear combinations is a linear combination, so $sp(S)$ is closed under vector addition and scalar multiplication. Hence $sp(S)$ is a subspace of V. [See text for details.] $\blacksquare$

The span of S is the smallest subspace containing S in the following sense:

Theorem: If S is a subset of V and W is any subspace of V containing S, then $sp(S) \subseteq W$.

$\boxed{\text{End Lec 4}}$