Proof: Since W contains S, the closure properties of W imply that W contains all linear combinations of vectors in S. Hence W contains $sp(S)$. ∎

Remarks:

① If S is itself a subspace of V, then $sp(S) = S$.

② By the Theorem, $sp(S)$ is the intersection of all subspaces containing S. [Exercise.]

Defn: We say that a subset S of V spans V (or is a spanning set for V) if $sp(S) = V$; that is, if every vector in V is a linear combination of vectors in S.

Problems and Examples

Our goal is to determine if a given vector $\bar{v}$ is in the span of a set $S = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$. The vector $\bar{v}$ is in $sp(S)$ if and only if there is a solution $\alpha_1, \ldots, \alpha_n$ ($\alpha_i \in K$) to the equation

$$\alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \cdots + \alpha_n \bar{v}_n = \bar{v} \quad (*).$$

In practice, the vector equation $(*)$ translates into a system of linear equations, and $\bar{v} \in sp(S)$ if and only if the system is consistent, i.e., a solution exists.

The set S spans V if and only if equation $(*)$ has a solution for every $\bar{v} \in V$. In terms of the associated system of linear equations, this means S spans V if and only if the system is solvable for all "right hand sides."

[See handouts on Echelon Form (Existence of Solutions) and Spanning & Linear Independence (Spanning Sets).]