EXAMPLES:

① $V = \mathbb{R}^3$, $S = \{(1,2,-1), (2,1,2), (4,5,0), (3,3,1)\}$.
Determine if the vector $\vec{v} = (2,3,4)$ is in $sp(S)$.

We need to determine whether the equation
$x(1,2,-1) + y(2,1,2) + z(4,5,0) + w(3,3,1) = (2,3,4)$
is solvable. Comparing coefficients, this translates to

$$\begin{array}{rrrrr} x & +2y & +4z & +3w & =2 \\ 2x & +y & +5z & +3w & =3 \\ -x & +2y & & w & =4 \end{array} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 5 & 3 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Reduce the augmented matrix to echelon form:

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 2 \\ 2 & 1 & 5 & 3 & 3 \\ -1 & 2 & 0 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 2 \\ 0 & -3 & -3 & -3 & -1 \\ 0 & 4 & 4 & 4 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 2 \\ 0 & 1 & 1 & 1 & 1/3 \\ 0 & 4 & 4 & 4 & 6 \end{array}\right]$$

Subtract 2x row 1 from row 2
Add row 1 to row 3

Divide row 2 by -3.

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 2 \\ 0 & 1 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 14/3 \end{array}\right]$$

Subtract 4x row 2 from row 3.

The last row corresponds to the equation $0x+0y+0z+0w = \frac{14}{3}$, which has no solution. Hence the system is inconsistent and no such x, y, z, w exist.

Hence $\vec{v} = (2,3,4)$ is not in $sp(S)$. □

* Recall that a system of equations is consistent if and only if in the echelon form of the augmented matrix, each zero row of the coefficient matrix corresponds to a 0 constant.