(EXAMPLES, cont.)

② Describe algebraically the vectors <u>in</u> $sp(S)$, where
$$S = \{(1,2,-1), (2,1,2), (4,5,0), (3,3,1)\} \subseteq \mathbb{R}^3$$
(as in Example 1).

A vector $(a,b,c) \in \mathbb{R}^3$ is in $sp(S)$ if and only if the equation
$$x(1,2,-1) + y(2,1,2) + z(4,5,0) + w(3,3,1) = (a,b,c)$$
is solvable. We reduce the augmented matrix of the equation to echelon form:

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 3 & a \\ 2 & 1 & 5 & 3 & b \\ -1 & 2 & 0 & 1 & c \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 4 & 3 & a \\ 0 & -3 & -3 & -3 & b-2a \\ 0 & 4 & 4 & 4 & a+c \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 4 & 3 & a \\ 0 & 1 & 1 & 1 & \frac{2a-b}{3} \\ 0 & 4 & 4 & 4 & a+c \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 4 & 3 & a \\ 0 & 1 & 1 & 1 & \frac{2a-b}{3} \\ 0 & 0 & 0 & 0 & -\frac{5}{3}a + \frac{4}{3}b + c \end{array}\right]$$

The system is solvable if and only if the zero row of the coefficient matrix corresponds to a zero constant. That is, $(a,b,c) \in sp(S)$ if and only if

$$(-5/3)a + \left(\tfrac{4}{3}\right)b + c = 0, \text{ or}$$

$$5a - 4b - 3c = 0. \text{ (equivalently, } c = \tfrac{5a-4b}{3}\text{)}.$$

Hence $sp(S) = \{(a,b,c) \in \mathbb{R}^3 \mid 5a - 4b - 3c = 0\}$.

Note that $(2,3,4)$ is <u>not</u> in $sp(S)$ as $5(2) - 4(3) - 3(4) = -14 \neq 0$. However, since $5(6) - 4(9) - 3(2) = 0$, we have $(6,9,2) \in sp(S)$.

$\boxed{\text{End Lec 5}}$ <u>Exercise</u>: Verify that each vector in S satisfies this equation.