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Lec 6
2/4/09
(EXAMPLES, cont.)

(3) Verify that $f(t) = 5 + t - 3t^2 \in P_2(t)$ is in the span of $S = \{1 - t + t^2,\ 2 + t - 2t^2,\ 3 - t^2\}$ and write $f(t)$ as a linear combination of vectors in S.

We need to solve the equation

$$x(1-t+t^2) + y(2+t-2t^2) + z(3-t^2) = 5 + t - 3t^2.$$

Equating coefficients yields the system of equations

$$\begin{aligned} x + 2y + 3z &= 5 \\ -x + y &= 1 \\ x - 2y - z &= -3 \end{aligned} \quad \longleftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ -1 & 1 & 0 & 1 \\ 1 & -2 & -1 & -3 \end{array}\right]$$

[notice the coefficients of elements of S are the <u>columns</u> of the matrix]

Reduce to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ -1 & 1 & 0 & 1 \\ 1 & -2 & -1 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 3 & 3 & 6 \\ 0 & -4 & -4 & -8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The zero coefficient row corresponds to a zero constant, hence the system is solvable. To write $f(t)$ as a combination of vectors in S, we solve:

$$\left.\begin{aligned} x + 2y + 3z &= 5 \\ y + z &= 2 \end{aligned}\right\}$$ so z is a free variable, $y = 2 - z$, and $x = 5 - 2y - 3z = 5 - 2(2-z) - 3z = 1 - z$.

Hence, for any z, the coefficients $x = 1 - z$, $y = 2 - z$, z work. For example, with $z = 0$, we have

$$\boxed{f(t) = 5 + t - 3t^2 = 1(1 - t + t^2) + 2(2 + t - 2t^2) + 0(3 - t^2)}$$

<u>Remarks</u>: (1) If the echelon form of the coefficient matrix has a zero row, the set S cannot span the entire space.

(2) In particular, <u>less than</u> n vectors cannot span $\mathbb{R}^n$. [Fewer than n pivots implies there will be a zero row.]