Linear Independence (§4.7)

As always, V is a vector space over K.

Defn: A subset S of V is linearly independent if for every set $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ of vectors in S, the only solution to $\alpha_1\bar{v}_1 + \cdots + \alpha_n\bar{v}_n = \bar{0}$ is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.
If there is a solution with some $\alpha_i \neq 0$, S is linearly dependent.

Note: Carefully read Remarks 1–6 (p 122) in the text. Most will be covered in class, but need to know all.

Theorem A subset S of V is linearly dependent if and only if some vector in S is a linear combination of other vectors in S.

Proof: $\boxed{\Rightarrow}$ If S is linearly dependent, then there are vectors $\bar{v}_1, \dots, \bar{v}_n$ and scalars $\alpha_1, \dots, \alpha_n$ with $\alpha_1\bar{v}_1 + \cdots + \alpha_n\bar{v}_n = \bar{0}$ and at least one $\alpha_i \neq 0$. We may assume without loss of generality that $\alpha_1 \neq 0$. Thus,

$$\bar{v}_1 = -\frac{\alpha_2}{\alpha_1}\bar{v}_2 - \frac{\alpha_3}{\alpha_1}\bar{v}_3 - \cdots - \frac{\alpha_n}{\alpha_1}\bar{v}_n,$$

and $\bar{v}_1$ is a linear combination of $\bar{v}_2, \dots, \bar{v}_n$.

$\boxed{\Leftarrow}$ Suppose a vector $\bar{v} \in S$ is a linear combination of $\bar{v}_1, \dots, \bar{v}_k$ in S, so that $\bar{v} = \alpha_1\bar{v}_1 + \cdots + \alpha_k\bar{v}_k$ for some $\alpha_i \in K$. Hence $\alpha_1 v_1 + \cdots + \alpha_k\bar{v}_k + (-1)\bar{v} = \bar{0}$, and at least one coefficient (-1) is nonzero. Thus S is linearly dependent. $\square$

Remarks: ① This says S is linearly independent if and only if no vector in S is a linear combination of other vectors in S.

② This can be refined to show that S is dependent if and only if some vector $\bar{v}_k \in S$ is a linear combination of $\bar{v}_1, \dots, \bar{v}_{k-1}$. [Lemma 4.10].

③ It is better to use the definition to prove independence most of the time. The theorem is useful for proving other results.