(EXAMPLES, cont.)

[② cont] The equation translates to the matrix equation

$$\begin{aligned} 2x + y + 4z &= 0 \\ -x + 2y + 3z &= 0 \\ 2x - y &= 0 \end{aligned} \quad \text{with coefficient matrix} \begin{bmatrix} 2 & 1 & 4 \\ -1 & 2 & 3 \\ 2 & -1 & 0 \end{bmatrix}.$$

[observe that the vectors in S form the columns of the coefficient matrix. Since the system is homogeneous, the augmented matrix is not needed.]

Reduce the coefficient matrix to echelon form:

$$\begin{bmatrix} 2 & 1 & 4 \\ -1 & 2 & 3 \\ 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 4 \\ 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & 3 \\ 0 & 5 & 10 \\ 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Since there is no pivot in the column corresponding to z, z is a free variable and the solution is <u>not</u> unique. Hence S is <u>linearly dependent</u>. ∎

③ Determine whether the set

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\} \subseteq M_2(\mathbb{R})$$

is linearly independent or linearly dependent.

We need to determine if the equation

$$x \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has a unique solution. This translates to the system

$$\begin{aligned} x + 2y + z &= 0 \\ x + y &= 0 \\ -y + z &= 0 \\ x + 3y + z &= 0 \end{aligned} \quad \text{with coefficient matrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since every column has a pivot, there are no free variables.

Thus, the system has a unique solution and S is <u>independent</u>. ∎