2

Defn: A <u>vector space</u> over a field K is a set V along with two operations – vector addition, scalar multiplication – satisfying the following axioms:

[C1] $\bar{u}+\bar{v} \in V$ for all $\bar{u}, \bar{v} \in V$ [Closure under vector addition]
[C2] $\alpha\bar{v} \in V$ for all $\alpha \in K$, $\bar{v} \in V$ [closure under scalar multiplication]

[A1] For all $\bar{u}, \bar{v}, \bar{w} \in V$, $\bar{u}+(\bar{v}+\bar{w}) = (\bar{u}+\bar{v})+\bar{w}$ [associative law]
[A2] There is a vector, denoted $\bar{0}$, in V such that $\bar{v}+\bar{0} = \bar{0}+\bar{v} = \bar{v}$ for all $\bar{v} \in V$ [zero vector]
[A3] For each $\bar{u} \in V$ there is a vector, denoted $-u$, such that $\bar{u}+(-\bar{u}) = (-\bar{u}+\bar{u}) = \bar{0}$ [additive inverse]
[A4] For all $\bar{u}, \bar{v} \in V$, $\bar{u}+\bar{v} = \bar{v}+\bar{u}$ [Commutative law]

[M1] For all $\alpha \in K$, $\bar{u}, \bar{v} \in V$, $\alpha(\bar{u}+\bar{v}) = \alpha\bar{u}+\alpha\bar{v}$
[M2] For all $\alpha, \beta \in K$, $\bar{v} \in V$, $(\alpha+\beta)\bar{v} = \alpha\bar{v}+\beta\bar{v}$ } [distributive laws]
[M3] For all $\alpha, \beta \in K$, $\bar{v} \in V$, $\alpha(\beta\bar{v}) = (\alpha\beta)\bar{v}$ [associative law]
[M4] For all $\bar{v} \in V$, $1 \in K$, $1\bar{v} = \bar{v}$.

<u>Remarks</u>:

1. [C1], [C2] are usually part of the definition of "operation." They are included here for emphasis.

2. Any set with operations satisfying these axioms is a vector space. Any general theorems derived from these axioms will hold in any specific examples.

3. The zero vector ($\bar{0}$) and additive inverses ($-\bar{v}$) depend on the set <u>and</u> the operations. The vector satisfying [A2] <u>is</u> the zero vector, even though it may not "look like" 0. The additive inverse of $\bar{v}$ is the vector $\bar{v}'$ satisfying $\bar{v}+\bar{v}' = \bar{0}$ (for the <u>correct</u> zero vector $\bar{0}$). [See HW1, Problem 2, for example.]

End Lec 1