(EXAMPLES, cont.)

(4) Is the set
$$S = \{(1,2,3), (5,17,12), (3,9,-42), (8,-7,6)\} \subseteq \mathbb{R}^3$$
linearly independent or linearly dependent?

The coefficient matrix of the corresponding homogeneous system of equations is
$$\begin{bmatrix} 1 & 5 & 3 & 8 \\ 2 & 17 & 9 & -7 \\ 3 & 12 & -42 & 6 \end{bmatrix}.$$
Since the echelon form can have at most 3 pivots (one in each row) and there are four columns, there must be a column without a pivot, hence a free variable. Thus S is linearly dependent.

Remark: A similar argument shows that any set of more than n vectors in $\mathbb{R}^n$ must be linearly dependent. The matrix will have n rows (hence at most n pivots) but more than n columns.

Basis and Dimension (§§ 4.8, 4.9)

Defn: A subset S of V is a basis for V if S is both linearly independent and spans V.

Theorem: If S is a basis for V, then every vector in V can be written in exactly one way as a linear combination of vectors in S.

Proof: Let $\bar{v}$ be a vector in V. Since S spans V, $\bar{v}$ can be written as a linear combination of vectors in S. Suppose $\bar{v}$ can written as a combination in two ways, say $\bar{v} = \alpha_1\bar{v}_1 + \cdots + \alpha_n\bar{v}_n = \beta_1\bar{v}_1 + \cdots + \beta_n\bar{v}_n$,