Proof: (i) If S is linearly independent then S is contained in a basis of n vectors, hence $|S| \leq n$.
(ii) If S spans V then S contains a basis of n vectors, hence $|S| \geq n$. $\square$

Corollary: Let $\dim V = n$ and let S be a set of exactly n vectors.
(i) If S spans V then S is a basis for V.
(ii) If S is linearly independent, then S is a basis for V.
That is, if $\dim V = n = |S|$, then S spans V if and only if S is linearly independent.

Proof: (i) If S spans V then S contains a basis B for V. But $|B| = n = |S|$, hence $S = B$.
(ii) If S is linearly independent, then S is contained in a basis B for V, and $|B| = n = S$. Hence $S = B$. $\square$

Theorem: Let W be a subspace of V. If B_W is a basis for W, then there is a basis B_V for V containing B_W.

Proof: A basis B_W for W is a linearly independent subset of V, so is contained in a basis B_V for V. $\square$

Corollary: Let V be a finite dimensional vector space, W a subspace of V. Then $\dim W \leq \dim V$, and $\dim W = \dim V$ if and only if $W = V$.

Proof: The first statement is immediate from the theorem. For the second, let B_W be a basis for W and B_V a basis for V containing B_W, so $|B_W| = \dim W \leq \dim V = |B_V|$. Thus $\dim V = \dim W$ if and only if $B_W = B_V$, and so if and only if $W = sp(B_W) = sp(B_V) = V$. $\square$

Note: The second statement in the corollary is false if V is infinite dimensional.