Basis Given a Spanning Set.

Recall: Let A be an $m \times n$ matrix.
The row space of A, $\mathcal{R}(A)$, is the span of the rows of A, a subspace of $\mathbb{R}^n_{row}$.
The columnspace of A, $\mathcal{C}(A)$, is the span of the columns of A, a subspace of $\mathbb{R}^m_{col}$.

Recall also that two matrices are row equivalent if one can be obtained from the other by elementary row operations. In particular, A is row equivalent to its (row) echelon form $\mathbf{E}$.

See §4.6 and §4.9 for justification of the following results.

Lemma: If A and B are row equivalent matrices, then
(i) $\mathcal{R}(A) = \mathcal{R}(B)$, i.e. A and B have the same rowspace;
(ii) $\mathcal{C}(A)$, $\mathcal{C}(B)$ are generally not the same, but a set of columns of A is linearly independent if and only if the set of corresponding columns of B is linearly independent.

Lemma: Let E be a matrix in (row) echelon form.
(i) The nonzero rows of E (i.e. the rows with pivots) form a basis for $\mathcal{R}(E)$, the row space of E.
(ii) the columns of E with pivots form a basis for $\mathcal{C}(E)$, the column space of E.

Theorem: Let E be the (row) echelon form of the matrix A.
(i) The nonzero rows of E form a basis for the row space $\mathcal{R}(A)$ of A.
(ii) The columns of A corresponding to the columns of E with pivots form a basis for the column space $\mathcal{C}(A)$ of A.