Example: Find a basis for $\mathcal{R}(A), \mathcal{C}(A)$ if $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 4 & 2 & 7 \\ 1 & 1 & 3 & 2 & 6 \\ 4 & 2 & 8 & 5 & 15 \end{bmatrix}$.

Reduce A to row echelon form:

$$A = \begin{array}{c} \begin{array}{ccccc} * & * & & * & \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 4 & 2 & 7 \\ 1 & 1 & 3 & 2 & 6 \\ 4 & 2 & 8 & 5 & 15 \end{bmatrix} \end{array} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{array}{c} \begin{array}{ccccc} * & * & & * & \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 1 & 1 & 2 \\ 0 & \boxed{1} & 2 & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = E$$

Thus $\mathcal{R}(A)$ has basis $\{(1,0,1,1,2), (0,1,2,0,3), (0,0,0,1,1)\}$
and $\mathcal{C}(A)$ has basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 5 \end{bmatrix} \right\}$.

Observe that both bases in the example have the same number of vectors, i.e. $\dim \mathcal{R}(A) = 3 = \dim \mathcal{C}(A)$, even though $\mathcal{R}(A) \subseteq \mathbb{R}^5$ and $\mathcal{C}(A) \subseteq \mathbb{R}^4$!

This is not a coincidence. By the theorem,

$\dim \mathcal{R}(A) = \#$ of nonzero rows of $E = \#$ of pivots of E,

$\boxed{\text{End Lec 8}}$ $\dim \mathcal{C}(A) = \#$ of columns of E with pivots $= \#$ of pivots of E.

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Corollary: If A is any matrix, then

$\dim \mathcal{R}(A) = \dim \mathcal{C}(A) = \#$ of pivots in the (row) echelon form of A.

Defn: If A is a matrix,

$\dim \mathcal{R}(A)$ is called the row rank of A,

$\dim \mathcal{C}(A)$ is called the column rank of A.

By the results above, the row rank and column rank are equal.

Defn: The rank of A is the number $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$.