Basis for Solution Space of a Homogeneous System

Consider a homogeneous system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

If $A = [a_{ij}]$ is the coefficient matrix, $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, and $\bar{0} = \bar{0}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then the system above is equivalent to the matrix equation $A\bar{x} = \bar{0}$.

Recall that the nullspace $\mathcal{N}(A) = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} = \bar{0}\}$ of A is a subspace of $\mathbb{R}^n$. The dimension of $\mathcal{N}(A)$ is the nullity of A.

The null space of the coefficient matrix A is equal to the solution space of the homogeneous system; that is, $\mathcal{N}(A)$ is equal to the set of all solutions $\bar{x}$ of the system.

Method to Determine Basis and Dimension of $\mathcal{N}(A)$

1. Reduce A to (row) echelon form E.
2. Solve for the pivot variables [corresponding to columns with pivots] in terms of the free variables [corresponding to columns without pivots].
3. The dimension of A is the number of free variables.
4. Construct a basis vector $\bar{x}$ for each free variable x_i by forming the solution vector with $x_i = 1$ and all other free variables equal to 0. The set of all such $\bar{x}$ forms a basis for $\mathcal{N}(A)$.

Theorem (Rank + Nullity theorem) If A is an $m \times n$ matrix of rank r, then the nullity of A is $n - r$; that is, $\text{rank}(A) + \text{nullity}(A) = \dim \mathcal{C}(A) + \dim \mathcal{N}(A) = n$.

Proof: We know $\dim \mathcal{C}(A)$ is the number of columns of E with pivots, $\dim \mathcal{N}(A)$ is the number of columns without pivots. $\square$