Lec 10, 2/13/09

Bases for Spaces Defined by Coordinates

EXAMPLE Let $U = \{(a,b,c,d) \in \mathbb{R}^4 \mid d = a+2b\}$
$W = \{(a,b,c,d) \in \mathbb{R}^4 \mid a = 2c,\ b = d\}$
Find bases for U, W, and $U \cap W$.

Basis for U: The variables a, b, c are free, so $\dim U = 3$.
(U is the solution space of the system $d - a - 2b = 0$.)
The method above gives the basis
$B_U = \{(1,0,0,1), (0,1,0,2), (0,0,1,0)\}$.

Basis for W: The variables c, d are free, so $\dim W = 2$.
(W is the solution space of the system
$$\begin{aligned} a \quad -2c \quad &= 0 \\ b \quad -d &= 0. \end{aligned}$$)

The method above gives basis
$B_W = \{(2,0,1,0), (0,1,0,1)\}$.

Basis for $U \cap W$: A vector (a,b,c,d) is in $U \cap W$ if and only if it satisfies all conditions defining U and W. Thus
$U \cap W = \{(a,b,c,d) \in \mathbb{R}^4 \mid d = a+2b,\ a = 2c,\ b = d\}$.
Thus $U \cap W$ is the solution space of the system

$$\begin{aligned} a + 2b \quad - d &= 0 \\ a \quad - 2c \quad &= 0, \text{ or } \mathcal{N}(A),\ A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \to \\ b \quad - d &= 0 \end{aligned}$$

$$\leftrightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -2 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

Hence d is free, $-2c - d = 0$, so $c = -\frac{1}{2}d$
$b - d = 0$ so $b = d$
and $a + 2b - d = 0$, so $a = d - 2b = d - 2d = -d$.
Then $\dim(U \cap W) = 1$ and $B_{U \cap W} = \{(-1, 1, -\frac{1}{2}, 1)\}$ is a basis.
[Note: $B_U \cap B_W$ is NOT a basis for $U \cap W$.]