

(30)

Sums and Direct Sums (§4.10)

Recall: (HW#2, Problem 6) If U, W are subspaces of a vector space V , then the sum of U and W is

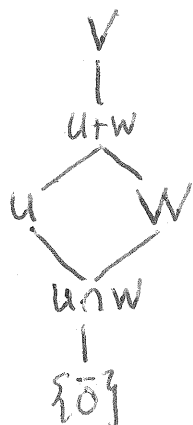
$$U+W = \{ \bar{u} + \bar{w} \mid \bar{u} \in U, \bar{w} \in W \},$$

and $U+W$ is also a subspace of V .

Remarks: ① U and W are both contained in $U+W$.

② If V' is any subspace of V containing both U and W , then V' contains $U+W$, by closure of V' under addition. Hence, $U+W =$ intersection of all subspaces containing U and W
 $=$ smallest subspace of V containing U and W .

③ By HW#4, Problem 1, if S is a spanning set for U and T a spanning set for W , then $S \cup T$ is a spanning set for $U+W$. In particular, if B_U, B_W is a basis for U, W , respectively, then $B_U \cup B_W$ is a spanning set for $U+W$ (not necessarily a basis).



Theorem: If U, W are finite dimensional subspaces of V , then $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$.

Sketch of Proof (See Problem 4.58.)

Let $\dim(U \cap W) = r$ and let $\{ \bar{v}_1, \dots, \bar{v}_r \}$ be a basis for $U \cap W$. Since $U \cap W$ is a subspace of U and of W , this can be extended to a basis $B_U = \{ \bar{v}_1, \dots, \bar{v}_r, \bar{u}_1, \dots, \bar{u}_m \}$ for U and to a basis $B_W = \{ \bar{v}_1, \dots, \bar{v}_r, \bar{w}_1, \dots, \bar{w}_n \}$ for W .

Show that $B = \{ \bar{v}_1, \dots, \bar{v}_r, \bar{u}_1, \dots, \bar{u}_m, \bar{w}_1, \dots, \bar{w}_n \}$ is a basis for $U+W$. Then

$$\begin{aligned} \dim U + \dim W - \dim(U \cap W) &= (m+r) + (n+r) - r \\ &= m+n+r = \dim(U+W). \end{aligned}$$

End Lec 10

□