Sums and Direct Sums (§4.10)

Recall: (HW#2, Problem 6) If U, W are subspaces of a vector space V, then the sum of U and W is
$$U+W = \{\bar{u}+\bar{w} \mid \bar{u}\in U,\ \bar{w}\in W\},$$
and $U+W$ is also a subspace of V.

$$\begin{tikzcd}
& V \arrow[d, dash] & \\
& U+W \arrow[dl, dash] \arrow[dr, dash] & \\
U \arrow[dr, dash] & & W \arrow[dl, dash] \\
& U\cap W \arrow[d, dash] & \\
& \{\bar{0}\} &
\end{tikzcd}$$

Remarks: ① U and W are both contained in $U+W$.

② If V' is any subspace of V containing both U and W, then V' contains $U+W$, by closure of V' under addition. Hence, $U+W$ = intersection of all subspaces containing U and W = smallest subspace of V containing U and W.

③ By HW#4, Problem 1, if S is a spanning set for U and T a spanning set for W, then $S\cup T$ is a spanning set for $U+W$. In particular, if B_U, B_W is a basis for U, W, respectively, then $B_U \cup B_W$ is a spanning set for $U+W$ (not necessarily a basis).

Theorem: If U, W are finite dimensional subspaces of V, then
$$\dim(U+W) = \dim U + \dim W - \dim(U\cap W).$$

Sketch of Proof (See Problem 4.58.)
Let $\dim(U\cap W) = r$ and let $\{\bar{v}_1, \dots, \bar{v}_r\}$ be a basis for $U\cap W$. Since $U\cap W$ is a subspace of U and of W, this can be extended to a basis $B_U = \{\bar{v}_1, \dots, \bar{v}_r, \bar{u}_1, \dots, u_m\}$ for U and to a basis $B_W = \{\bar{v}_1, \dots, \bar{v}_r, w_1, \dots, \bar{w}_n\}$ for W.
Show that $B = \{\bar{v}_1, \dots, \bar{v}_r, \bar{u}_1, \dots, \bar{u}_m, \bar{w}_1, \dots, \bar{w}_n\}$ is a basis for $U+W$. Then
$$\dim U + \dim W - \dim(U\cap W) = (m+r)+(n+r)-r$$
$$= m+n+r = \dim(U+W). \quad \square$$

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