Direct Sums

<u>Defn</u>: Let U, W be subspaces of V. We say V is the <u>direct sum</u> of U and W, denoted $U \oplus W$, if
(i) $V = U + W$
(ii) $U \cap W = \{\bar{0}\}$.

The following theorem is used as the <u>definition</u> of direct sum in the text, with the conditions above stated as a theorem.

<u>Theorem</u>: If $V = U \oplus W$, then every vector in V can be written in exactly one way as the sum of a vector in U and a vector in W.

<u>Proof</u>: Let $V = U \oplus W$, so $V = U + W$ and $U \cap W = \{\bar{0}\}$.
For $\bar{v} \in V$, we can write $\bar{v} = \bar{u} + \bar{w}$, $\bar{u} \in U$, $\bar{w} \in W$, since $\bar{v} \in V = U + W$.
Suppose $\bar{v} = \bar{u} + \bar{w}$ and $\bar{v} = \bar{u}' + \bar{w}'$, with $\bar{u}, \bar{u}' \in U$, $\bar{w}, \bar{w}' \in W$. Then $\bar{u} + \bar{w} = \bar{u}' + \bar{w}'$, and so $\bar{u} - \bar{u}' = \bar{w}' - \bar{w}$. Since U, W are subspaces, $\bar{u} - \bar{u}' \in U$ and $\bar{w}' - \bar{w} \in W$, hence $\bar{u} - \bar{u}' = \bar{w}' - \bar{w} \in U \cap W = \{\bar{0}\}$. That is, $\bar{u} - \bar{u}' = \bar{0}$ and $\bar{w}' - \bar{w} = \bar{0}$, and so $\bar{u} = \bar{u}'$, $\bar{w} = \bar{w}'$, and the expression is unique. $\square$

See text for the definition and basic results on <u>general</u> direct sums (Theorems 4.22 and 4.23). The following result is not in the text.

<u>Theorem</u>: If V is a finite dimensional vector space and U is a subspace of V, then there is a subspace W of V such that $V = U \oplus W$

[The subspace W is called a <u>complement</u> of U in V, and is <u>not</u> unique.]