③ By Note ②, a vector space over K of dimension n is "algebraically" the same as K^n. We say V is isomorphic to K^n.

EXAMPLES

① Let $V = P_n(t)$ and $B = \{1, t, t^2, \dots, t^n\}$ the standard basis. Then

$$[\alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n]_B = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^{n+1}$$

Polynomials in $P_n(t)$ add by adding corresponding coefficients and scalar multiplication is termwise, just as in $\mathbb{R}^{n+1}$.

② Let $V = M_2(\mathbb{R})$ and $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

hence $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \longleftrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$

For example, if $\bar{w} = \begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ then $[\bar{w}]_B = \begin{bmatrix} 2 \\ -3 \\ 5 \\ 7 \end{bmatrix}$.

③ Let $V = P_2(t)$ and $B = \{1+t, 2+t, t+t^2\}$ (Check B is a basis.) Find the coordinate vector $[\bar{v}]_B$ for $\bar{v} = a + bt + ct^2$.

We need to solve the equation $x(1+t) + y(2+t) + z(t+t^2) = a + bt + ct^2$ for x, y, z (in terms of a, b, c). Equating coefficients yields:

$$\begin{aligned} x + 2y \quad &= a \\ x + y + z &= b \\ z &= c \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 1 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 0 & 1 & c \end{array}\right] \rightarrow \begin{aligned} x + 2y \quad &= a \\ -y + z &= b - a \\ z &= c \end{aligned}$$

Solving the last system we have $\underline{z = c}$, $y = z + a - b = \underline{a - b + c = y}$ and $x = a - 2y = a - 2a + 2b - 2c = \underline{-a + 2b - 2c = x}$.

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Hence $[a + bt + ct^2]_B = \begin{bmatrix} -a + 2b - c \\ a - b + c \\ c \end{bmatrix}$. For example, $[1 + 2t + 3t^2]_B = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$.