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Inner Product Spaces (§7.2, §7.3)

For now, V will always be a vector space over $\mathbb{R}$

Recall that in $\mathbb{R}^2$, $\mathbb{R}^3$, the dot product can be used to define distances and angles. We generalize this by defining an inner product on V.

Defn: Let V be a vector space over $\mathbb{R}$. To each ordered pair $\bar{u}, \bar{v}$ of vectors in V, associate a real number $\langle \bar{u}, \bar{v} \rangle$. This function is a (real) inner product if it satisfies:

[Linear] (I_1) (a) $\langle \bar{u}_1 + \bar{u}_2, \bar{v} \rangle = \langle \bar{u}_1, \bar{v} \rangle + \langle \bar{u}_2, \bar{v} \rangle$ for all $\bar{u}_1, \bar{u}_2, \bar{v} \in V$

(b) $\langle \alpha \bar{u}, \bar{v} \rangle = \alpha \langle \bar{u}, \bar{v} \rangle$ for all $\alpha \in \mathbb{R}$, $\bar{u}, \bar{v} \in V$.

[Symmetric] (I_2) $\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$ for all $\bar{u}, \bar{v} \in V$.

[Positive Definite] (I_3) $\langle \bar{u}, \bar{u} \rangle \geq 0$ for all $\bar{u} \in V$ and $\langle \bar{u}, \bar{u} \rangle = 0$ if and only if $\bar{u} = \bar{0}$.

A vecter space V on which an inner product is defined is called an inner product space.

Remarks:

① (I_1) says that $\langle \cdot, \cdot \rangle$ is linear in the first variable. By symmetry (I_2), it is also linear in the second:

$$\langle \bar{u}, \bar{v}_1 + \bar{v}_2 \rangle = \langle \bar{u}, \bar{v}_1 \rangle + \langle u, \bar{v}_2 \rangle,$$
$$\langle \bar{u}, \alpha \bar{v} \rangle = \alpha \langle \bar{u}, \bar{v} \rangle.$$

Hence $\langle \cdot, \cdot \rangle$ is bilinear.

② Using linearity in both variables, we have in general

$$\langle \alpha_1 \bar{u}_1 + \cdots + \alpha_m \bar{u}_m, \beta_1 \bar{v}_1 + \cdots + \beta_n \bar{v}_n \rangle = \sum_{j=1}^{n} \sum_{i=1}^{m} \alpha_i \beta_j \langle \bar{u}_i, \bar{v}_j \rangle.$$

③ If $\bar{v}$ is any vector, then $\langle \bar{0}, \bar{v} \rangle = \langle \bar{v}, \bar{0} \rangle = 0$.

(To see this, note that $\langle \bar{0}, \bar{v} \rangle = \langle 0\bar{0}, \bar{v} \rangle = 0 \langle \bar{0}, \bar{v} \rangle = 0$.)