EXAMPLES

① The "standard" inner product on $\mathbb{R}^n$ generalizes the dot product on $\mathbb{R}^2, \mathbb{R}^3$. For $\bar{u} = (\alpha_1, \cdots, \alpha_n)$, $\bar{v} = (\beta_1, \cdots, \beta_n)$, we define

$$\langle \bar{u}, \bar{v} \rangle = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n \quad (= \bar{u} \cdot \bar{v})$$

[Exercise: Verify I_1, I_2, I_3.]

Note: Unless otherwise stated, the standard inner product will be used when discussing $\mathbb{R}^n$! This IS NOT the only inner product on $\mathbb{R}^n$, however. For example:

② Let $V = \mathbb{R}^2$. We define another inner product as follows: For $\bar{u} = (x_1, x_2)$ and $\bar{v} = (y_1, y_2)$ in $\mathbb{R}^2$, define

$$\langle \bar{u}, \bar{v} \rangle = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2.$$

That is is an inner product can be verified easily. The verification of I_3 is a little tricky:
if $\bar{u} = (x_1, x_2)$ then

$$\langle \bar{u}, \bar{u} \rangle = x_1x_1 - 2x_1x_2 - 2x_2x_1 + 5x_2x_2 = x_1^2 - 4x_1x_2 + 5x_2^2$$
$$= x_1^2 - 4x_1x_2 + 4x_2^2 + x_2^2 = (x_1 - 2x_2)^2 + x_2^2 \geq 0.$$

(here's the tricky part) Hence $\langle \bar{u}, \bar{u} \rangle \geq 0$.

Moreover, $\langle \bar{u}, \bar{u} \rangle = (x_1 - 2x_2)^2 + x_2^2 = 0$ if and only if $x_1 - 2x_2 = 0$ and $x_2 = 0$, hence also $x_1 = 0$. So $\langle \bar{u}, \bar{u} \rangle = 0 \Leftrightarrow \bar{u} = \bar{0}$.
(The verification of I_1, I_2 are exercises.)

Sample Calculation:

$$\langle (1,2), (3,-1) \rangle = (1)(3) - 2(1)(-1) - 2(2)(3) + 5(2)(-1)$$
$$= 3 + 2 - 12 - 10 = -17.$$

③ Let $V = P_n(t)$. For $f(t) = a_0 + a_1t + \cdots + a_nt^n$, $g(t) = b_0 + b_1t + \cdots + b_nt^n$, define $\langle f(t), g(t) \rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$.

Observe that relative to the basis $B = \{1, t, t^2, \cdots, t^n\}$, we have $[f(t)]_B = (a_0, a_1, \cdots, a_n)$, $[g(t)]_B = (b_0, b_1, \cdots, b_n)$ and $\langle f(t), g(t) \rangle = [f(t)]_B \cdot [g(t)]_B$.
We generalize this in the next example.