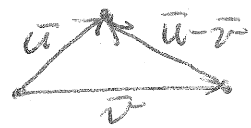
(38)

Distance and Angles

In $\mathbb{R}^2$ and $\mathbb{R}^3$, we define the distance between two vectors (with "tails" at the origin) to be the distance between the terminal points of the vectors:



This is the length of the vector $\bar{u}-\bar{v}$. We generalize this accordingly:

Defn: If V is a real inner product space, the <u>distance</u> between vectors $\bar{u},\bar{v}\in V$ is $d(\bar{u},\bar{v}) = \|\bar{u}-\bar{v}\|$.

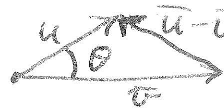
Observe that by property [N2],

$$d(\bar{v},\bar{u}) = \|\bar{v}-\bar{u}\| = \|(-1)(\bar{u}-\bar{v})\| = |-1|\,\|\bar{u}-\bar{v}\| = d(\bar{u},\bar{v}).$$

So the distance from $\bar{u}$ to $\bar{v}$ is the same as from $\bar{v}$ to $\bar{u}$.

Recall also that in $\mathbb{R}^2$ and $\mathbb{R}^3$, the angle θ between vectors $\bar{u},\bar{v}$ is the angle θ with $0\le\theta\le\pi$ satisfying

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\,\|\bar{v}\|}.$$



This is derived using the Law of Cosines:

$$\|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\|\bar{u}\|\,\|\bar{v}\|\cos\theta = \|\bar{u}-\bar{v}\|^2 =$$
$$= (u-v)\cdot(u-\bar{v}) = \bar{u}\cdot\bar{u} - 2\bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{v}$$
$$= \|\bar{u}\|^2 - 2\bar{u}\cdot\bar{v} + \|\bar{v}\|^2.$$

Thus $-2\|\bar{u}\|\,\|\bar{v}\|\cos\theta = -2\,\bar{u}\cdot\bar{v}$ and the formula follows.

More generally, by the Cauchy-Schwarz inequality, we have $|\langle\bar{u},\bar{v}\rangle| \le \|\bar{u}\|\,\|\bar{v}\|$, hence $-1 \le \dfrac{\langle\bar{u},\bar{v}\rangle}{\|\bar{u}\|\,\|\bar{v}\|} \le 1$. We define

Defn: If V is a real inner product space, $\bar{u},\bar{v}\in V$ <u>nonzero</u> vectors, the <u>angle</u> θ between $\bar{u}$ and $\bar{v}$ is the angle θ, $0\le\theta\le\pi$, with $\cos\theta = \dfrac{\langle\bar{u},\bar{v}\rangle}{\|\bar{u}\|\,\|\bar{v}\|}$