Orthogonality (§7.5)

By the definition of the angle between vectors, if $\bar{u}, \bar{v}$ are nonzero vectors then the angle θ is $\frac{\pi}{2}$ (or $90°$) if and only if $\cos\theta = 0$, hence if and only if $\langle \bar{u}, \bar{v} \rangle = 0$.

This motivates the following definition:

Defn: Let V be an inner product space. Vectors $\bar{u}, \bar{v}$ in V are orthogonal if $\langle \bar{u}, \bar{v} \rangle = 0$.

Remarks: ① This is slightly more general than "perpendicular". Note that $\bar{u}, \bar{v}$ need not be nonzero vectors. In particular, as $\langle \bar{0}, \bar{v} \rangle = 0$ for all $\bar{v} \in V$, $\bar{0}$ is orthogonal to every vector in V.

② Orthogonality is symmetric; that is $\bar{u}$ is orthogonal to $\bar{v}$ if and only if $\bar{v}$ is orthogonal to $\bar{u}$.

Defn: If V is an inner product space and S is a nonempty subset of V, the orthogonal complement of S is

$$S^{\perp} = \{ \bar{v} \in V \mid \langle \bar{v}, \bar{s} \rangle = 0 \text{ for all } \bar{s} \in S \}.$$

(That is, $S^{\perp}$ is the set of vectors in V that are orthogonal to every vector in S.)

Theorem: If S is a nonempty subset of an inner product space V, then $S^{\perp}$ is a subspace of V.

Proof: (i) As noted above, $\langle \bar{0}, \bar{s} \rangle = 0$ for all $\bar{s} \in S$, hence $\bar{0} \in S^{\perp}$ and $S^{\perp} \neq \emptyset$.
(ii) If $\bar{u}, \bar{v} \in S^{\perp}$ then $\langle \bar{u}, \bar{s} \rangle = \langle \bar{v}, \bar{s} \rangle = 0$ for all $\bar{s} \in S$, hence $\langle \bar{u} + \bar{v}, \bar{s} \rangle = \langle \bar{u}, \bar{s} \rangle + \langle \bar{v}, \bar{s} \rangle = 0 + 0 = 0$ for all $\bar{s} \in S$, hence $\bar{u} + \bar{v} \in S^{\perp}$.
(iii) If $\alpha \in \mathbb{R}$, $\bar{v} \in S^{\perp}$, so $\langle \bar{v}, \bar{s} \rangle = 0$ for all $\bar{s} \in S$, then $\langle \alpha\bar{v}, \bar{s} \rangle = \alpha \langle \bar{v}, \bar{s} \rangle = 0$ for all $\bar{s} \in S$. Hence $\alpha \bar{v} \in S^{\perp}$. □

[END LEC 13]