(40)

Lec 14, 2/23/09

EXAMPLES:

① Recall that $\langle \bar{0}, \bar{v} \rangle = 0$ for all $\bar{v}$ in an inner product space V. It follows that $\underline{\{\bar{0}\}^\perp = V}$.

② If V is any inner product space and $\bar{u} \in V^\perp$, then $\langle \bar{u}, \bar{v} \rangle = 0$ for all $\bar{v} \in V$. In particular, $\langle \bar{u}, \bar{u} \rangle = 0$, so by $[I_3]$, $\bar{u} = \bar{0}$. Hence $\underline{V^\perp = \{\bar{0}\}}$.

③ Let $V = \mathbb{R}^3$ with the standard inner product, and let $S = \{(1,2,-1), (2,-1,2)\}$. Then

$$S^\perp = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y,z)\cdot(1,2,-1) = 0, (x,y,z)\cdot(2,-1,2) = 0\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid x+2y-z=0,\ 2x-y+2z=0\}$$
$$= \mathcal{N}(A), \text{ where}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 4 \end{bmatrix} = E \qquad \begin{matrix} x+2y-z=0 \\ -5y+4z=0 \end{matrix}$$

Thus z is free, $y = \frac{4}{5}z$, $x = z - 2y = z - \frac{8}{5}z = -\frac{3}{5}z$.

Hence

$$S^\perp = \{(-\tfrac{3}{5}z, \tfrac{4}{5}z, z) \mid z \in \mathbb{R}\} = \{(-3r, 4r, 5r) \mid r \in \mathbb{R}\},$$

and $\{(-3,4,5)\}$ is a basis for $S^\perp$, $\dim S^\perp = 1$.

④ Let $V = \mathbb{R}^n$ with the standard inner product, and let A be a fixed $m \times n$ matrix, so $\mathcal{N}(A) \subseteq \mathbb{R}^n$. Then

$x \in \mathcal{N}(A) \iff A\bar{x} = \bar{0}$

$\iff$ The dot product of each $\underline{\text{row}}$ of A with $\bar{x}$ is 0

$\iff \bar{x}$ is orthogonal to each row of A

$\iff \bar{x} \in \mathcal{R}(A)^\perp$

Hence (ignoring the distinction between row vectors in R(A) and column vectors in $\mathcal{N}(A)$) we have

$$\mathcal{R}(A)^\perp = \mathcal{N}(A).$$

<u>Exercise:</u> Show that the <u>left</u> null space of A is the orthogonal complement of the column space of A.