Remarks: the following are easy to prove (see Problem 7.14):

① If $S_1 \subseteq S_2$, then $S_2^\perp \subseteq S_1^\perp$.

② $S^\perp = (sp(S))^\perp$, hence if S is a spanning set for a subspace W, then $S^\perp = W^\perp$.

③ $S \subseteq (S^\perp)^\perp$.

The following result will be proved later (Theorems 7.9, 7.10).

Theorem: If W is a subspace of a finite dimensional inner product space V, then there is a basis $B_W = \{\bar{w}_1, \dots, \bar{w}_m\}$ for W with $\langle \bar{w}_i, \bar{w}_j \rangle = 0$ for all $i \neq j$. Moreover, B_W can be extended to a basis $B_V = \{\bar{w}_1, \dots, \bar{w}_m, \bar{w}_{m+1}, \dots, \bar{w}_n\}$ for V with the same property.

Lemma If W is a subspace of an inner product space V, then $W \cap W^\perp = \{\bar{0}\}$.

Proof: Since W and $W^\perp$ are subspaces, certainly $\bar{0} \in W \cap W^\perp$. Conversely, if $\bar{w} \in W \cap W^\perp$, then in particular $\bar{w} \in W^\perp$ and so is orthogonal to every vector in W, including $\bar{w}$. Thus $\langle \bar{w}, \bar{w} \rangle = 0$, and so $\bar{w} = \bar{0}$ by $[I_3]$. Hence $W \cap W^\perp = \{\bar{0}\}$. ∎

Theorem If W is a subspace of a finite dimensional inner product space, then $V = W \oplus W^\perp$.

Proof: By the Lemma, $W \cap W^\perp = \{\bar{0}\}$, so it remains to show $V = W + W^\perp$. By the theorem, there is a basis $B_V = \{\bar{w}_1, \dots, \bar{w}_m, \bar{w}_{m+1}, \dots, \bar{w}_n\}$ with $\langle \bar{w}_i, \bar{w}_j \rangle = 0$ for all $i \neq j$, $B_W = \{\bar{w}_1, \dots, \bar{w}_m\}$ a basis for W. If $\bar{v} \in V$, then $\bar{v} = (\alpha_1 \bar{w}_1 + \dots + \alpha_m \bar{w}_m) + (\alpha_{m+1} \bar{w}_{m+1} + \dots + \alpha_n \bar{w}_n)$, some $\alpha_i \in \mathbb{R}$. Thus $\alpha_1 \bar{w}_1 + \dots + \alpha_m \bar{w}_m \in W$ and $\alpha_{m+1} \bar{w}_{m+1} + \dots + \alpha_n \bar{w}_n \in W^\perp$ [Exercise], and so $\bar{v} \in W + W^\perp$. ∎

END LEC 14