[Proof of Theorem, cont.]

Then, for each $i = 1, \cdots, n$,

$$0 = \langle \bar{v}_i, \bar{0} \rangle = \langle \bar{v}_i, \alpha_1 \bar{v}_1 + \cdots + \alpha_n \bar{v}_n \rangle$$
$$= \alpha_1 \langle \bar{v}_i, \bar{v}_1 \rangle + \cdots + \alpha_i \langle \bar{v}_i, \bar{v}_i \rangle + \cdots + \alpha_n \langle \bar{v}_n, \bar{v}_n \rangle$$
$$= \alpha_i \langle \bar{v}_i, \bar{v}_i \rangle, \text{ since } S \text{ is orthogonal.}$$

Now since $\bar{v}_i \neq \bar{0}$, we have $\langle \bar{v}_i, \bar{v}_i \rangle \neq 0$, and so $\alpha_i \langle \bar{v}_i, \bar{v}_i \rangle = 0$ implies $\alpha_i = 0$. This holds for each i, hence S is linearly independent. $\square$

Note: This implies that if $\dim V = n$ then any orthogonal set of n nonzero vectors is a basis for V.

One reason that orthogonal bases are useful is that it is easy to find the coordinates of a vector relative to an orthogonal basis.

Theorem: Let $B = \{\bar{u}_1, \cdots, \bar{u}_n\}$ be an orthogonal basis for V. For $\bar{v} \in V$,

$$\bar{v} = \frac{\langle \bar{v}, \bar{u}_1 \rangle}{\langle \bar{u}_1, \bar{u}_1 \rangle} \bar{u}_1 + \cdots + \frac{\langle \bar{v}, \bar{u}_n \rangle}{\langle \bar{u}_n, \bar{u}_n \rangle} \bar{u}_n;$$

That is, the i^{th} coordinate of $\bar{v}$ relative to B is $\frac{\langle \bar{v}, \bar{u}_i \rangle}{\langle \bar{u}_i, \bar{u}_i \rangle}$.

If B is an orthonormal basis, then

$$\bar{v} = \langle \bar{v}, \bar{u}_1 \rangle \bar{u}_1 + \cdots + \langle \bar{v}, \bar{u}_n \rangle \bar{u}_n.$$

Proof Outline [See Problem 7.17] Write $\bar{v} = \alpha_1 \bar{u}_1 + \cdots + \alpha_n \bar{u}_n$ and compute $\langle \bar{v}, \bar{u}_i \rangle$ for each i. Obtain $\langle \bar{v}, \bar{u}_i \rangle = \alpha_i \langle \bar{u}_i, \bar{u}_i \rangle$. $\square$

EXAMPLE: If $V = \mathbb{R}^3$ with standard inner product, then $S = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$, where $\bar{u}_1 = (1,1,1)$, $\bar{u}_2 = (1,1,-2)$, $\bar{u}_3 = (-1,1,0)$, is an orthogonal basis.

By the Theorem, if $\bar{v} = (2,-1,3)$, then

$$\bar{v} = \frac{\langle \bar{v}, \bar{u}_1 \rangle}{\langle \bar{u}_1, \bar{u}_1 \rangle} \bar{u}_1 + \frac{\langle \bar{v}, \bar{u}_2 \rangle}{\langle \bar{u}_2, \bar{u}_2 \rangle} \bar{u}_2 + \frac{\langle \bar{v}, \bar{u}_3 \rangle}{\langle \bar{u}_3, \bar{u}_3 \rangle} \bar{u}_3$$

$$= \frac{2-1+3}{1+1+1} \bar{u}_1 + \frac{2-1-6}{1+1+4} \bar{u}_2 + \frac{-2-1}{1+1} \bar{u}_3 = \frac{4}{3}\bar{u}_1 - \frac{5}{6}\bar{u}_2 - \frac{3}{2}\bar{u}_3$$

Hence $[(2,-1,3)]_S = \begin{bmatrix} 4/3 \\ -5/6 \\ -3/2 \end{bmatrix}$. (Check.)