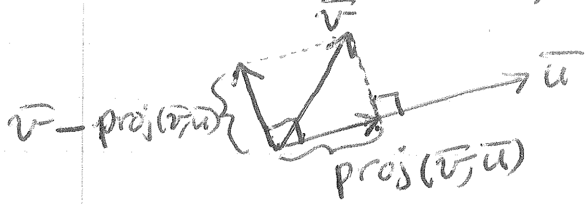
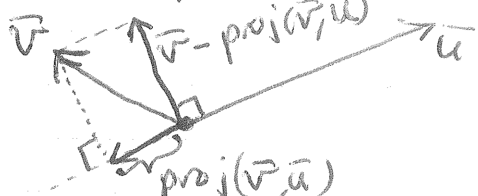
Projections

Given vectors $\bar{u}, \bar{v}$ in $\mathbb{R}^2$, we "project" $\bar{v}$ onto $\bar{u}$:





The projection, $\text{proj}(\bar{v},\bar{u})$ satisfies:

- $\text{proj}(\bar{v};\bar{u}) = c\bar{u}$ for some scalar c
- $\bar{v}-c\bar{u}$ is orthogonal to $\bar{u}$.
- $c\bar{u}$ is the "closest" vector to $\bar{v}$ in the span of $\bar{u}$.

We generalize this to any inner product space V over $\mathbb{R}$.

Lemma: Let $\bar{u},\bar{v} \in V$, $\bar{u} \neq \bar{0}$. Then $c = \dfrac{\langle \bar{v},\bar{u}\rangle}{\langle \bar{u},\bar{u}\rangle}$ is the unique scalar c such that $\bar{v}-c\bar{u}$ is orthogonal to $\bar{u}$.

Proof: We have $\bar{v}-c\bar{u}$ is orthogonal to $\bar{u}$ for a scalar c if and only if

$$(*) \quad 0 = \langle \bar{v}-c\bar{u}, \bar{u}\rangle = \langle \bar{v},\bar{u}\rangle - c\langle \bar{u},\bar{u}\rangle.$$

Since $\bar{u} \neq \bar{0}$, we have $\langle \bar{u},\bar{u}\rangle \neq \bar{0}$, so equation $(*)$ is equivalent to $c = \langle \bar{v},\bar{u}\rangle / \langle \bar{u},\bar{u}\rangle$ as claimed. $\square$

Defn: For $\bar{u},\bar{v} \in V$, $\bar{u} \neq \bar{0}$, the projection of $\bar{v}$ along $\bar{u}$ is the vector

$$\text{proj}(\bar{v};\bar{u}) = \frac{\langle \bar{v};\bar{u}\rangle}{\langle \bar{u},\bar{u}\rangle}\bar{u}.$$

The scalar $c = \dfrac{\langle \bar{v},\bar{u}\rangle}{\langle \bar{u},\bar{u}\rangle}$ is the component of $\bar{v}$ along $\bar{u}$ or the Fourier Coefficient of $\bar{v}$ with respect to $\bar{u}$.

$\boxed{\text{End Lec 15}}$

Lec 16, 3/2/09

EXAMPLE: Let $V = \mathbb{R}^3$ (standard inner product), $\bar{v} = (-3,1,2)$ and $\bar{u} = (-1,2,1)$. Then

$$\text{proj}(\bar{v};\bar{u}) = \frac{\langle \bar{v},\bar{u}\rangle}{\langle \bar{u},\bar{u}\rangle}\bar{u} = \frac{3+2+2}{1+4+1}\bar{u} = \frac{7}{6}\bar{u} = \left(-\frac{7}{6}, \frac{7}{3}, \frac{7}{6}\right).$$