Note: In the example, we have
$$\bar{v} - \mathrm{proj}(\bar{v},\bar{u}) = (-3,1,2) - \left(-\tfrac{7}{6}, \tfrac{7}{3}, \tfrac{7}{6}\right) = \left(-\tfrac{11}{6}, -\tfrac{4}{3}, \tfrac{5}{6}\right),$$
and
$$\langle \bar{u}, \bar{v} - \mathrm{proj}(\bar{v},\bar{u})\rangle = \left\langle (-1,2,1), \left(-\tfrac{11}{6}, -\tfrac{4}{3}, \tfrac{5}{6}\right)\right\rangle = \tfrac{11}{6} - \tfrac{8}{3} + \tfrac{5}{6} = 0,$$
as expected by the lemma. Thus we have
$$\bar{v} = \underbrace{\left(-\tfrac{7}{6}, \tfrac{7}{3}, \tfrac{7}{6}\right)}_{\text{parallel to } \bar{u}} + \underbrace{\left(-\tfrac{11}{6}, -\tfrac{4}{3}, \tfrac{5}{6}\right)}_{\text{orthogonal to } \bar{u}}.$$

Proposition: If $\bar{u}, \bar{v} \in V$, $\bar{u} \neq \bar{0}$, then $\mathrm{proj}(\bar{v};\bar{u})$ is the closest vector to $\bar{v}$ in the span of $\bar{u}$; that is, for all $\bar{\omega} \in sp(\bar{u})$,
$$\|\bar{v} - \mathrm{proj}(\bar{v};\bar{u})\| \leq \|\bar{v} - \bar{\omega}\|.$$

Proof: If $\bar{\omega} \in sp(\bar{u})$, then $\bar{\omega} = \alpha\bar{u}$ for some $\alpha \in \mathbb{R}$.
Let $c = \langle \bar{v}, \bar{u}\rangle / \langle \bar{u}, \bar{u}\rangle$, so $\mathrm{proj}(\bar{v};\bar{u}) = c\bar{u}$, and then
$$\begin{aligned}
\|\bar{v} - \alpha\bar{u}\|^2 &= \langle \bar{v} - \alpha\bar{u}, \bar{v} - \alpha\bar{u}\rangle = \\
&= \langle (\bar{v} - c\bar{u}) + (c-\alpha)\bar{u}, (\bar{v} - c\bar{u}) + (c-\alpha)\bar{u}\rangle \\
&= \langle \bar{v} - c\bar{u}, \bar{v} - c\bar{u}\rangle + 2(c-\alpha)\overset{0}{\langle \bar{v} - c\bar{u}, \bar{u}\rangle} + (c-\alpha)^2\langle \bar{u}, \bar{u}\rangle \\
&= \|\bar{v} - c\bar{u}\|^2 + \underbrace{(c-\alpha)^2\|\bar{u}\|^2}_{\geq 0} \geq \|\bar{v} - c\bar{u}\|^2. \quad \square
\end{aligned}$$

In particular, note that $\mathrm{proj}(\bar{v};\bar{u}) = \bar{v}$ if and only if $\bar{v} \in sp(\bar{u})$.

We can generalize the projection onto $sp(\bar{u})$ to any subspace of V.

Defn: Let W be a subspace of V. If $B = \{\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_k\}$ is an ORTHOGONAL BASIS for W, the projection of a vector $\bar{v} \in V$ onto W is
$$\mathrm{proj}(\bar{v}; W) = \frac{\langle \bar{v}, \bar{\omega}_1\rangle}{\langle \bar{\omega}_1, \bar{\omega}_1\rangle}\bar{\omega}_1 + \dots + \frac{\langle \bar{v}, \bar{\omega}_k\rangle}{\langle \bar{\omega}_k, \bar{\omega}_k\rangle}\bar{\omega}_k;$$
that is, $\mathrm{proj}(\bar{v}, W)$ is the sum of the projections of $\bar{v}$ along the basis vectors $\bar{\omega}_i$.

Note: This yields a vector with the desired properties only if B is an ORTHOGONAL basis for W.